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PSEUDOLINEARITY AND EFFICIENCY VIA DINI DERIVATIVES

SHASHI AGGARWAL

Department of Mathematics, Miranda House, University of Delhi, Delhi 110007

AND

DAVINDER BHATIA

Department of Mathematics, S.G.T.B. Khalsa College, University of Delhi, Delhi 110007

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In this paper, charaterization of pseudolinear functions in terms of Dini derivatives is given. Necessary and sufficient conditions for efficiency in terms of Dini derivatives are derived for multiobjective programming problems involving pseudolinear functions.

INTRODUCTION

Nonlinear multiobjective programming problems involving pseudolinear functions have been studied by Chew and Choo[1]. They have characterized pseudolinear functions by means of proportional functions under the assumption of differentiability. Rockafellar[5] has pointed out that the functions involved may not be always differentiable and so characterization of pseudolinear functions without assuming differentiability is required.

Kaul *et al.*[4] have defined semilocally pseudolinear functions in terms of right derivatives generalizing pseudolinear functions and have extended results of Chew and Choo[1] concerning efficiency for multiobjective programming problems involving functions that are semilocally pseudolinear. It was shown be Kaul *et al.*[4] that a function f is semilocally pseudolinear on a set $\Gamma \subseteq R^n$ iff $\exists$ a real valued function p called proportional function of f, defined on $\Gamma \times \Gamma$ such that $p(x, y) > 0$ and $f(y) = f(x) + p(x, y)\, df^+(x, y-x)$ for and x, y in Γ.

The purpose of this paper is to define pseudolinear functions in terms of Dini derivatives[2,3] which generalize the class of semilocally pseudolinear functions. In this paper, we consider the multiobjective pseudolinear programming problem of the form

$$\text{(P)} \quad \operatorname{Max} f(x) = (f_1(x), \ldots, f_k(x))$$

subject to

$$x \in X = \{x \in S \mid g_j(x) \geqslant 0,\ j = 1, \ldots, m\}$$

where it is assumed that f_i, $i = 1,\dots,k$ and g_j, $j = 1,\dots,m$ are real pseudolinear functions defined on a convex subset S of R^N.

Here an alternative characterization of pseudolinearity is provided which makes use of Dini Derivatives instead of right derivatives. The useful feature about Dini derivatives is that they always exists whereas right derivatives need not. Such a characterization seems necessary because of the existence of functions of the type given in the example presented in section 2, which motivated the authors to make a present study.

Theorems 2 and 3 develop necessary and sufficient conditions for the existence of an efficient solution for the above mentioned problem (P), whereas Kaul *et al.*[4] have derived only sufficient condition.

2. Dini Derivatives and Pseudolinear Functions

Definition 1 – Let f be a real valued function defined over S, a convex subset of R^N. Let $x \in R^N$, $v \in R^N$ with $v^T v = 1$. Dini derivatives of f in the direction v at x are defined as follows :

$$D_v^{+u} f(x) = \lim_{n\to\infty} \sup_{\{t_n\}} \left\{ \frac{f(x + t_n v) - f(x)}{t_n} : 0 < t_n \leqslant 1/n \right\}$$

$$D_v^{+l} f(x) = \lim_{n\to\infty} \inf_{\{t_n\}} \left\{ \frac{f(x + t_n v) - f(x)}{t_n} : 0 < t_n \leqslant 1/n \right\}$$

$$D_v^{-u} f(x) = \lim_{n\to\infty} \sup_{\{t_n\}} \left\{ \frac{f(x - t_n v) - f(x)}{-t_n} : 0 < t_n \leqslant 1/n \right\}$$

$$D_v^{-l} f(x) = \lim_{n\to\infty} \inf_{\{t_n\}} \left\{ \frac{f(x - t_n v) - f(x)}{-t_n} : 0 < t_n \leqslant 1/n \right\}.$$

Here $D_v^{+u} f(x)$ is the upper right derivative, $D_v^{+l} f(x)$ is the lower right derivative, $D_v^{-u} f(x)$ is the upper left derivative and $D_v^{-l} f(x)$ is lower left derivative evaluated at x in the direction v. Limits can be infinite in the above definition. It may easily be proved, by using the definitions, that

(I) Dini derivatives always exist (finite or infinite) for any function f and satisfy

$$D_v^{+u} f(x) > D_v^{+l} f(x),\ D_v^{-u} f(x) \geqslant D_v^{-l} f(x)$$

$$D_v^{+u} (-f)(x) = -\ D_v^{+l} f(x),\ D_v^{+l} (-f)(x) = -\ D_v^{+u} f(x)$$

(II) If $D_v^{+l} f(x) = D_v^{+u} f(x)$ (or if $D_v^{-l} f(x) = D_v^{-u} f(x)$) then the common value, written $D_v^{+} f(x)$ (or $D_v^{-} f(x)$) is just the right (or left) derivative of f at x in the

direction v. In general, $D_v^- f(x) \leqslant D_v^+ f(x)$ and if $D_v^- f(x) = D_v^+ f(x)$, then f has the derivative at x in the direction v and the common value will be denoted by $D_v f(x)$.

If f is a function of one variable, then usually, we take the directional vector v to be the scalar 1 and we write

$$D_1^{+u} f(x) \equiv D^{+u} f(x), \quad D_1^{+l} f(x) \equiv D^{+l} f(x)$$

$$D_1^{+} f(x) \equiv D^{+} f(x), \quad D_1^{-} f(x) \equiv D^{-} f(x).$$

Diewert[2] introduces pseudoconcave functions as :

Definition 2—A function f defined over a convex subset S of R^N is said to be pseudoconcave over S iff for every $x^\circ \in S$, $v \in R^N$ satisfying

$$v^T v = 1, \ t > 0, \ x^\circ + t\, v \in S, \ D_v^{+u} f(x^\circ) \leqslant 0$$

implies

$$f(x^\circ + t\, v) \leqslant f(x^\circ).$$

In the light of the above definition 2, we may now define pseudoconvex functions as follows :

Definition 3—A function f defined over a convex subset S of R^N is said to be pseudoconvex over S iff $-f$ is pseudoconcave over S i. e. for every $x^\circ \in S$, $v \in R^N$ satisfying

$v^T v = 1, t > 0, \ x^\circ + t\, v \in S, \ D_v^{+l} f(x^\circ) \geqslant 0$ implies $f(x^\circ + t\, v) \geqslant f(x^\circ)$.

Hence a pseudolinear function in terms of Dini derivatatives may be defined as follows

Definition 4—A function f defined over a convex subset S of R^N is said to be pseudolinear over S iff it is both pseudoconcave and pseudoconvex according to definitions 2 and 3 above.

Example—Let

$$f(x) = \begin{cases} 1 & , \ -2 \leqslant x \leqslant -1 \\ 0 & , \ -1 < x \leqslant 0 \\ 1/2^{n+1} & , \ 1/2^{n+1} \leqslant x < 1/2^n, \ n = 0, 1, 2, \ldots \end{cases}$$

be a function defined over $[-2, 1[$.

Some of the Dini derivatives computed at different points of the domain of the function are as follows :

$$D^{+u} f(-1) = -\infty, \quad D^{+l} f(-1) = -\infty$$

$$D^{+u} f(0) = 1, \quad D^{+l} f(0) = 1/2.$$

It may be easily verified that

$$D^{+u} f(-1) < 0 \Rightarrow f(-1+t) < f(-1), t > 0, (-1+t) \in]-1, 0]$$

$$D^{+l} f(0) > 0 \Rightarrow f(0+t) > f(0), t > 0, 0+t \in]0, 1[.$$

Thus f is pseudolinear according to definition 4 over $[-2, 1[$. It is obvious that f does not have the right derivative at $x = 0$ and hence f is not semilocally pseudolinear Kaul *et al.*[4] over $[-2, 1]$.

Assumption—In the sequel, we assume all the functions and Dini derivatives to be finite.

Theorem 1—Let f be a function defined over a convex subset S of R^N. Then the following statements are equivalent.

(i) f is pseudolinear over S.

(ii) There exist real functions p and q defined over $S \times S$ such that $p(x^\circ, x^\circ + tv) > 0$, $q(x^\circ, x^\circ + tv) > 0$ and

$$f(x^\circ + tv) = f(x^\circ) + p(x^\circ, x^\circ + tv)\, D_v^{+u} f(x^\circ) + q(x^\circ, x^\circ + tv)\, D_v^{+l} f(x^\circ) \quad \ldots(1)$$

for any $x^\circ \in S$, $v \in R^N$ satisfying $v^T v = 1$, $t > 0$, $x^\circ + tv \in S$.

PROOF : (i) $\Rightarrow$ (ii). Let $x^\circ \in S$, $v \in R^N$ satisfying $v^T v = 1$, $t > 0$, $x^\circ + tv \in S$ and f be pseudolinear over S. Therefore

$$\left.\begin{array}{l} \text{if } D_v^{+u} f(x^\circ) \leqslant 0 \text{ then } f(x^\circ + tv) \leqslant f(x^\circ) \\ \text{and if } D_v^{+l} f(x^\circ) \geqslant 0 \text{ then } f(x^\circ + tv) \geqslant f(x^\circ) \end{array}\right] \quad \ldots(2)$$

As the right derivative of f may not exist at every point of S, we have two cases.

Case 1: When right derivative at x° does not exist, i. e.

$$D_v^{+u} f(x^\circ) \neq D_v^{+u} f(x^\circ)$$

Case 2: When right derivative at x° exists i. e.

$$D_v^{+} f(x^\circ) = D_v^{+u} f(x^\circ) = D_v^{+l} f(x^\circ).$$

Hence in case (1), relation (2) leads to

(a) $D_v^{+u} f(x^\circ) < 0 \Rightarrow D_v^{+l} f(x^\circ) < 0$ and $f(x^\circ + tv) < f(x^\circ)$

(b) $D_v^{+u} f(x^\circ) = 0 \Rightarrow D_v^{+l} f(x^\circ) < 0$ and $f(x^\circ + tv) < f(x^\circ)$

(c) $D_v^{+l} f(x^\circ) > 0 \Rightarrow D_v^{+u} f(x^\circ) > 0$ and $f(x^\circ + tv) > f(x^\circ)$

(d) $D_v^{+l} f(x^\circ) = 0 \Rightarrow D_v^{+u} f(x^\circ) > 0$ and $f(x^\circ + tv) > f(x^\circ)$

since $D_v^{+l} f(x) \leqslant D_v^{+u} f(x)$ for all $x \in S$ and moreover for any $x \in S$, $v \in R^N$ satisfying $v^T v = 1$, $t > 0$, $x + tv \in S$ if $f(x + tv) = f(x)$ then $D_v^{+u} f(x) = D_v^{+l} f(x) = 0$.

Now we establish statement (ii) in each of the above four possibilities. In possibilities (a) and (c) above we may define

$$p(x^\circ, x^\circ + tv) = \frac{f(x^\circ + tv) - f(x^\circ)}{D_v^{+u} f(x^\circ)}$$

and

$$q(x^\circ, x^\circ + tv) = \frac{f(x^\circ + tv) - f(x^\circ)}{D_v^{\cdot l} f(x^\circ)}$$

and the results follows.

In case of possibility (b), we can define $p(x^\circ, x^\circ + tv)$ to be any positive real number and

$$q(x^\circ, x^\circ + tv) = \frac{f(x^\circ + tv) - f(x^\circ)}{D_v^{+l} f(x^\circ)}.$$

Similarly, in case of possibility (d), we define

$$p(x^\circ, x^\circ + tv) = \frac{f(x^\circ + tv) - f(x^\circ)}{D_v^{+u} f(x^\circ)}$$

and $q(x^\circ, x^\circ + tv)$ to be any positive real number. Thus in all the four possibilities, statement (ii) holds.

In case 2, relation (1) can be written as

$$f(x^\circ + tv) = f(x^\circ) + p'(x^\circ, x^\circ + tv)\; D_v^{+} f(x^\circ)$$

where $p'(x^\circ, x^\circ + tv) = p(x^\circ, x^\circ + tv) + q(x^\circ, x^\circ + tv)$ and can be defined as

$$p'(x^\circ, x^\circ + tv) = \begin{cases} \dfrac{f(x^\circ + tv) - f(x^\circ)}{D_v^+ f(x^\circ)} & \text{if } D_v^+ f(x^\circ) \neq 0 \\ K, \text{ a positive real number, if } D_v^+ f(x^\circ) = 0. \end{cases}$$

The positiveness of the function p' can be proved exactly in the same manner as proved for the functions p and q in Case 1.

(ii) $\Rightarrow$ (i) : Suppose that there are functions p and q defined over $S \times S$ such that $p(x^\circ, x^\circ + tv) > 0$, $q(x^\circ, x^\circ + tv) > 0$ and relation (1) holds.

Then $D_v^{+u} f(x^\circ) \leqslant 0$ and the fact that $D_v^{+l} f(x^\circ) < D_v^{+u} f(x^\circ)$ imyly that $f(x^\circ + tv) \leqslant f(x^\circ)$ showing that f is pseudoconcave over S. Also $D_v^{+l} f(x^\circ) \geqslant 0$ and again the fact that $D_v^{+u} f(x^\circ) > D_v^{+l} f(x^\circ)$ imply that $f(x^\circ + tv) \geqslant f(x^\circ)$ showing that f is pseudoconvex over S.

Hence f is pseudolinear over S.

Q.E.D.

3. Necessary and Sufficient Conditions for the Existence of Efficient Solution

Throughout this section f_i, $i = 1, \ldots, k$, g_j, $j = 1, \ldots, m$ will be real pseudolinear functions according to definition 4 over a convex subset S of R^N with positive functions $p_i, q_i, i = 1, \ldots, k$ and p_j', q_j', $j = 1, \ldots, m$ respectively.

Consider the following multiobjective pseudolinear programming problem :

$$\text{(P)} \quad \operatorname{Max} f(x) = (f_1(x), \ldots, f_k(x))$$

subject to

$$x \in X = \{x \in S \mid g_j(x) \geqslant 0,\ j = 1, 2, \ldots, m\}.$$

For a point x in X, we shall denote by $I(x)$, the set of all j such that $g_j(x) = 0$

Definition 5—A vector $v \in R^N$ satisfying $v^T v = 1$ is called a feasible direction for X at $x^* \in X$ if there exists a $\delta > 0$ such that $x^* + tv \in X$ for all $0 < t \leqslant \delta$.

The set of all feasible directions at $x^* \in X$ is denoted by $D(x^*)$ and

$$D(x^*) = \{v \in R^N : v^T v = 1, \exists\ \delta > 0 \text{ such that } x^* + tv \in X \text{ for } 0 < t \leqslant \delta\}.$$

Definition 6—A point $x^* \in X$ is said to be efficient solution of problem (P) if there exists no $v \in R^N$ with $v^T v = 1$ such that

$$f_i(x^* + tv) \geqslant f_i(x^*),\ i = 1, \ldots, k$$

and $\quad f_i(x^* + t v) > f_i(x^*)$ for at least one i,

where $0 < t \leqslant \delta$ for some $\delta > 0$ such that $x^* + tv \in X$.

Note 1 : For $x^* \in X$, denote

$$p_i^* = p_i(x^*, x^* + t^* v),\ q^*_i = q_i(x^*, x^* + t^* v),\ i = 1, \dots, k$$

$$p'^*_j = p'_j(x^*, x^* + t^* v),\ q'^*_j = q'_j(x^*, x^* + t^* v),\ j = 1, \dots, m$$

where we can define[2]

$$v = \frac{(x - x^*),}{[(x - x^*)^T (x - x^*)]^{1/2}},\ x = x^* + t^* v \in X \text{ for}$$

$$t^* = [(x - x^*)^T (x - x^*)]^{1/2} > 0.$$

Lemma—Let x^* be a feasible solution for problem (P), then

$$p'^*_j D_v^{+u} g_j(x^*) + q'^*_j D_v^{+l} g_j(x^*) > 0 \text{ for } j \in I(x^*)$$

and

$$p'^*_j D_v^{+u} g_j(x^*) + q'^*_j D_v^{+l} g_j(x^*) > -\infty,\ j \notin I(x^*) \text{ implies } v \in D(x^*).$$

PROOF : Let $j \in I(x^*)$ and $p'^*_j D_v^{+u} g_j(x^*) + q'^*_j D_v^{+l} g_j(x^*) > 0$ for some direction v. Then $g_j(x^*) = 0$ since $j \in I(x^*)$. Suppose that there exists $\delta_j > 0$ such that $0 < t_j \leqslant \delta_j$ and $g_j(x^* + t_j v) < 0$. Then

$$p'^*_j D_v^{+u} g_j(x^*) + q'^*_j D_v^{+l} g_j(x^*)$$

$$= p'^*_j \limsup_{t \to 0} \frac{g_j(x^* + t v) - 0}{t} + q'^*_j \liminf_{t \to 0^+} \frac{g_j(x^* + tv) - 0}{t} \leqslant 0$$

which contradicts $p'^*_j D_v^{+u} g_j(x^*) + q^*_j D_v^{+l} g_j(x^*) > 0$.

Thus there exists some $\delta_j > 0$ such that

$$g_j(x^* + t v) \geqslant 0 \text{ for } 0 < t \leqslant \delta_j,\ j \in I(x^*). \qquad \dots(3)$$

Now let $j \notin I(x^*)$ and $p'^*_j D_v^{+u} g_j(x^*) + g'^*_j D_j^{+l} g_j(x^*) > -\infty$ for some direction v. Since $j \notin I(x^*)$, therefore $g_j(x^*) > 0$. Suppose that there exists $\delta_j > 0$ such that $0 < t_j \leqslant \delta_j$ and $g_j(x^* + t_j v) < 0$. Then

$$p'^*_j D_v^{+u} g_j(x^*) + q'^*_j D_v^{+l} g_j(x^*) = p'^*_j \limsup_{t \to 0^+}$$

(*equation continued on p.* 1180)

$$+ \frac{g_j(x^* + tv) - g_j(x^*)}{t} + q_j'^* \liminf_{t \to 0^+} \frac{g_j(x^* + tv) - g_j(x^*)}{t}$$

$$\to -\infty$$

which contradicts $p_j'^* D_v^{+u} g_j(x^*) + q_j'^* D_v^{+l} g_j(x^*) > -\infty$.

Thus there exists some $\delta_j > 0$ such that

$$g_j(x^* + tv) \geqslant 0,\ 0 < t \leqslant \delta_j,\ j \notin I(x^*) \qquad \ldots(4)$$

(3) and (4) imply that $v \in D(x^*)$. Q.E.D.

Theorem 2—If x^* is an efficient solution of problem (P) and constraints are assumed to satisfy conditions of Lemma then there exists $\lambda_i > 0$, $i = 1, \ldots, k$ and $\mu_j \geqslant 0$, $j \in I(x^*)$ such that

$$\sum_{i=1}^{k} \lambda_i [p_i^* D_v^{+u} f_i(x^*) + q_i^* D_v^{+l} f_i(x^*)]$$

$$+ \sum_{j \in I(x^*)} \mu_j [p_j'^* D_v^{+u} g_j(x^*) + q_j'^* D_v^{+l} g_j(x^*)] = 0. \qquad \ldots(5)$$

PROOF : Let x^* be an efficient solution of problem (P) and constraints satisfy the conditions of Lemma. Then by Lemma, $v \in D(x^*)$ and thus there is a $\delta > 0$ such that $x^* + tv \in X$ for $0 < t \leqslant \delta$.

We now assert that for $1 \leqslant r \leqslant k$ the system

$$(A) \quad \begin{cases} [p_j'^* D_v^{+u} g_j(x^*)\, v + q_j'^* D_v^{+l} g_j(x^*)\, v]^T (x^* + tv - x^*) \geqslant 0, \\ \qquad\qquad j \in I(x^*) & \ldots(6) \\ [p_i^* D_v^{+u} f_i(x^*)\, v + q_i^* D_v^{+l} f_i(x^*)\, v]^T (x^* + tv - x^*) \geqslant 0, \\ \qquad\qquad i = 1, \ldots, k,\ i \neq r & \ldots(7) \\ [p_r^* D_v^{+u} f_r(x^*)\, v + q_r^* D_v^{+l} f_r(x^*)\, v]^T (x^* + tv - x^*) > 0 \\ & \ldots(8) \end{cases}$$

has no solution $x^* + tv \in S$ for $0 < t \leqslant \delta$.

Let, if possible, $x^* + tv \in S$ for some $t \in]0, \delta]$ be a solution of the system A. Now as f_i's are pseudolinear over S with positive functions p_i's and q_i's, therefore relation (1) yields

$$f_i(x^* + tv) - f_i(x^*) = p_i(x^*, x^* + tv)\, D_v^{+u} f_i(x^*)$$

$$+ q_i(x^*, x^* + tv)\, D_v^{+l} f_i(x^*)\ i = 1, \ldots, k \qquad \ldots(9)$$

and clearly $x = x^* + t\,v \in X$.

As in Note 1, we may now define

$$v = \frac{x - x^*}{[(x - x^*)^T (x - x^*)]^{1/2}},$$

$$x = x^* + t^* v \text{ for } t^* = [(x - x^*)^T (x - x^*)]^{1/2} > 0.$$

Hence (9) implies

$$f_i(x) - f_i(x^*) = p_i^* \; D_v^{+u} \; f_i(x^*) + q_i^* \; D_v^{+l} \; f_i(x^*), \; i = 1, \ldots, k$$

$$\geqslant 0, \; i = 1, \ldots, k, \; i \neq r \text{ (using (7))}. \qquad \ldots(10)$$

Similarly $f_r(x) - f_r(x^*) > 0$ (using (8)). ...(11)

(10) and (11) contradict that x^* is efficient for problem (P). Hence system A has no solution $x^* + tv \in S$ for $0 < t \leqslant \delta$. Therefore, by Farkas' Lemma given by Mangasarian[5], there exist $\lambda_{r_i} \geqslant 0$, $\mu_{r_j} \geqslant 0$, $r = 1, \ldots, k$ such that

$$\sum_{j \in I(x^*)} \mu_{rj} \; [p_j'^* \; D_v^{+u} g_j(x^*) \, v + q_j'^* \; D_v^{+l} \; g_j(x^*) \, v]^T$$

$$+ \sum_{\substack{i=1 \\ i \neq r}}^{k} \lambda_{r_i} \; [p_i^* \; D_v^{+u} f_i(x^*) \, v + q_i^* \; D_v^{+l} f_i(x^*) \, v]^T$$

$$- \; [p_r^* \; D_v^{+u} \; f_r(x^*) \, v \; + q_r^* \; D_v^{+l} f_r(x^*) \, v]^T = 0. \quad \ldots(12)$$

Summing (12) over $r = 1, 2, \ldots, k$, we get

$$\sum_{i=1}^{k} \lambda_i \; [p_i^* \; D_v^{+u} f_i(x^*) \, v + q_i'^* D^{+l}_v f_i(x^*) v]^T$$

$$\sum_{j \in I(x^*)} \mu_j \, [p_j^{r*} \, D_v^{+u} \; g_j(x^*) \, v \; + q_j'^* \; D_v^{+l} \; g_j(x^*) \, v]^T = 0 \qquad \ldots(13)$$

where

$$\lambda_i = 1 + \sum_{r \neq i} \lambda_{ri} > 0, \; \mu_j = \sum_{r=1}^{k} \mu_{rj} \geqslant 0.$$

Relation (13) yields relation (5). Q.E.D.

Theorem 3—Let x^* be a feasible solution of the problem (P) and let there exist $\lambda_i > 0$, $i = 1, \ldots, k$, $\mu_j \geqslant 0$, $j \in I(x^*)$ such that relation (5) holds. Then x^* is efficient for problem (P).

PROOF : Let, if possible, x^* be not an efficient solution of problem (P). Then there exists some $v \in R^N$ with $v^T v = 1$ such that

$$\left.\begin{array}{l} f_i(x^* + tv) \geqslant f_i(x^*),\ i = 1, \ldots, k,\ i \neq r \\ f_r(x^* + tv) > f_r(x^*) \\ g_j(x^* + tv) \geqslant 0,\ j = 1, \ldots, m \end{array}\right] \quad \ldots(14)$$

where $0 < t \leqslant \delta$ for some $\delta > 0$ such that $x^* + tv \in S$.

Further on using relation (1) and defining p_i^*, q_i^*, $p_j'^*$, $q_j'^*$ as in Note 1, relation (14) gives

$$\sum_{i=1}^{k} \lambda_i (p_i^* \ D_v^{+u} f_i(x^*) + q_i^* \ D_v^{+l} f_i(x^*)) > 0$$

and

$$p_j'^* \ D_v^{+u} g_j(x^*) + q_j'^* \ D_v^{+l} g_j(x^*) \geqslant 0,\ j \in I(x^*)$$

Hence

$$0 \leqslant \sum_{j \in I(x^*)} [p_j'^* \ D_v^{+u} g_j(x^*) + q_j'^* \ D_v^{+l} g_j(x^*)]$$

$$= -\sum_{i=1}^{k} \lambda_i [p_i^* \ D_v^{+u} f_i(x^*) + q_i'^* \ D_v^{+l} f_i(x^*)]$$

$$< 0$$

which is an obvious contradiction.

Hence x^* is efficient for problem (P).

Q.E.D.

Note 2 : More general results where in problem (P) the objective functions are pseudoconcave and constraints are quasiconcave using Dini derivatives are developed in authors subsequent paper.

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ON THE EXISTENCE OF UNITY IN LEHMER'S ψ-PRODUCT RING

V. SITARAMAIAH

Department of Mathematics, Pondicherry Engineering College
Pillaichavady, Pondichey 605104

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Let T be a non-empty subset of $Z^+ \times Z^+$ where Z^+ is the set of positive integers and $\psi : T \to Z^+$ be a mapping such that for each $n \in Z^+$, $\psi(x,y) = n$ has a finite number of solutions. If f and g are two arithmetic functions, then the binary operation ψ on the set of arithmetic function F is defined by

$$(f \psi g)(n) = \sum_{\psi(x,y)=n} f(x)\, g(y)$$

for any $n \in Z^+$. If $(F, +, \psi)$ is a commutative ring, where $+$ denotes the usual pointwise addition and $\psi(x, y) \geqslant \max\{x, y\}$ for all $(x, y) \in T$, we prove that the ring $(F, +, \psi)$ possesses the unity if and only if for each $k \in Z^+$, $\psi(x, k) = k$ has a solution. In such a case the unity can be explicitly determined.

INTRODUCTION

An arithmetic function is a complex-valued function whose domain is the set of positive integers Z^+. Let F denote the set of all arithmetic functions. Let T be a non-empty subset of $Z^+ \times Z^+$ and let $\psi : T \to Z^+$ be a mapping satisfying the following postulates:

(I) For each $n \in Z^+$, $\psi(x, y) = n$ has a finite number of solutions.

(II) If $(x, y) \in T$, then $(y, x) \in T$ and $\psi(x, y) = \psi(y, x)$.

(III) $(y, z) \in T$ and $(x, \psi(y, z)) \in T$ if and only if "$(x, y) \in T$ and $(\psi(x, y), z) \in T$"; whenever one of these conditions holds, we have $\psi(x, \psi(y, z)) = \psi(\psi(x, y), z)$.

(IV) $\psi(1, 1) = 1$ and for each $k \in Z^+$, $\psi(x, k) = k$ has a solution.

(V) For each $k \in Z^+$, $k = \max\{x \in Z^+ : \psi(x, y) = k$ for some $y \in Z^+\}$ or equivalently $\psi(x, y) \geqslant \max\{x, y\}$ for all $(x, y) \in T$.

If we define the binary operation ψ on F by

$$(f \psi g)(n) = \sum_{\psi(x,y)=n} f(x)\, g(y)$$

for each $n \in Z^+$, $f, g \in F$, then using the postulates (I), (II) and (III) it is easily seen that $(F, +, \psi)$ is a commutative ring.

The following is proved in Sita Ramaiah[1] :

Lemma 1.1 (Sita Ramaiah[1], Lemma 2.1)—An arithmetic function g is an identity with respect to ψ if and only if, for any fixed $k, n \in Z^+$

$$\sum_{\psi(x,k)=n} g(x) = \begin{cases} 1, & \text{if } n = k \\ 0 & \text{if } n \neq k. \end{cases} \qquad \ldots(1.1)$$

Let

$$S_k = \{x : \psi(x, k) = k\} \qquad \ldots(1.2)$$

$$i_k = \min S_k \qquad \ldots(1.3)$$

and

$$S = \{i_k : k = 1, 2, \ldots\}. \qquad \ldots(1.4)$$

The number i_k in (1.3) exists by postulate (IV).

If the equation $\psi(x, k) = n$ has a unique solution in S if $n = k$ and no solution in S if $n = k$, it was mentioned in Theorem 2 of (Sita Ramaiah[1]) that the characteristic function X of S would be the unity of the commutative ring $(F, +, \psi)$.

It was proved in Sita Ramaiah[2] (Theorem 3.1) that if g is the unity of $(F, +, \psi)$, then g is integer-valued and $g(i_k) = 1$, where i_k is as given in (1.3). In addition, if g is non-negative, then g must be the characteristic function of the set S given in (1.4) (See Sita Ramaiah[2], Theorem 3.2). Also, an example of a ring $(F, +, \psi)$ was given (Sita Ramaiah[2], Example 3.1) in which the unity assumed negative-values also.

Suppose g is the unity of $(F, +, \psi)$. We shall now investigate the question of determining g from the relation (1.1). From (1.1), we have

$$\sum_{x \in S_k} g(x) = 1$$

so that for $k \in S_k$

$$g(k) = 1 - \sum_{\substack{x<k \\ x \in S_k}} g(x) \qquad \ldots(1.5)$$

where S_k is as given in (1.2). Let $k \notin S_k$. Let x_r be the largest element in S_k so that $x_r < k$. From (1.1), we obtain

$$\sum_{\psi(x,x_r)=k} g(x) = 0. \qquad \ldots(1.6)$$

Now, $x_r \in S_k$ implies that $\psi(k, x_r) = k$. Hence from (1.6),

$$g(k) = - \sum_{\substack{\psi(x,x_r)=k \\ x<k}} g(x). \qquad \ldots(1.7)$$

If $g(1) = 1$ and $g(x)$ has been defined for $1 \leqslant x < k$, then the relations (1.5) and (1.7) completely determine the value of $g(k)$ for any $k \in Z^+$. The task is to prove the otherway. We show that (See Lemma 2 2) the relation (1.7) implies that $g(k) = 0$ whenever $k \notin S_k$. Using this and (1.5) we ultimately show that (see Theorem 2.1) g is the unity of $(F, +, \psi)$, thus establishing the existence of the unity in the ring $(F, +, \psi)$, in the presence of the postulates (I) through (V).

2. Main Results

First we prove the following :

Lemma 2.1—We have

(i) $a, b \in S_k \Rightarrow \psi(a, b) \in S_k$.

(ii) If $a \in S_k$, then $S_a \subseteq S_k$.

(iii) $\psi(a, b) = k$ implies that $S_a \subseteq S_k$ and $S_b \subseteq S_k$.

(iv) $S_k = S_{x_r}$, where x_r is the largest element in S_k.

Proof : (i) $k = \psi(a, k) = \psi(a, \psi(b, k)) = \psi(\psi(a, b), k)$.

(ii) If $x \in S_a$, then $\psi(x, a) = a$. Hence

$$\psi(x, k) = \psi(x, \psi(a, k)) = \psi(\psi(x, a), k) = \psi(a, k) = k,$$

so that $x \in S_k$.

(iii) Let $\psi(a, b) = k$. Let $x \in S_a$. Then $\psi(x, a) = a$. Hence

$$\psi(x, k) = \psi(x, \psi(a, b)) = \psi(\psi(x, a), b) = \psi(a, b) = k.$$

Hence $x \in S_k$ so that $S_a \subseteq S_k$. Similarly $S_b \subseteq S_k$.

(iv) If $k \in S_k$, then $x_r = k$. So we may assume that $k \notin S_k$. Hence $x_r < k$. Since $x_r \in S_k$, by (ii), $S_{x_r} \subseteq S_k$. Let $x \in S_k$. Then $\psi(x, x_r) \in S_k$ by (i). Also, $\psi(x, x_r) \geqslant x_r$. Since x_r is the largest element in S_k, $\psi(x, x_r) = x_r$. Hence $x \in S_{x_r}$, implying that $S_k \subseteq S_{x_r}$.

Lemma 2.2—If g is the unity of $(F, +, \psi)$, then $g(k) = 0$ whenever $k \notin S_k$.

Proof : If $k \in S_k$ for every k, then there is nothing to prove. We assume that $k \notin S_k$ for some k. Let t be the least positive integer such that $t \notin S_t$ clearly $t \geqslant 2$ since $1 \in S_1$. For $1 \leqslant j < t$, $j \in S_j$. We have from (1.7),

$$g(t) = - \sum_{\substack{\psi(x, x_r) = t \\ x < t}} g(x)$$

where x_r is the largest in S_t. Now $x < t$ implies that $x \in S_x$. Also, by (iii) of Lemma 2.1, $\psi(x, x_r) = t$ implies that $S_x \subseteq S_t$. Hence $x \in S_t$. Since $x_r \in S_t$,

(i) of Lemma 2.1 implies that $t = \psi(x, x_r) \in S_t$ and this is false. Hence the sum on the right-hand side defining $g(t)$ is an empty sum. Hence $g(t) = 0$. We assume that $g(x) = 0$ whenever $x \notin S_x$ and $t \leqslant x < k$.

Let $k \notin S_k$. By (1.7), we have

$$g(k) = - \sum_{\substack{\psi(x, x_r) = k \\ x < k}} g(x)$$

where x_r is the largest element in S_k. If $1 \leqslant x < t$ and $\psi(x, x_r) = k$, then $x \in S_x \subseteq S_k$, implying that $k = \psi(x, x_r) \in S_k$. We may assume that $x \geqslant t$. For $t \leqslant x < k$, if $x \notin S_x$, $g(x) = 0$, by our induction hypothesis. Arguing as before we obtain

$$g(k) = - \sum_{\substack{\psi(x, x_r) = k \\ t \leqslant x < k \\ x \in S_x}} g(x) = \text{empty sum} = 0$$

since $k \notin S_k$. We now prove

Theorem 2.1—Let g be defined by

$$g(k) = \begin{cases} 1 - \displaystyle\sum_{\substack{x < k \\ x \in S_k}} g(x), & \text{if } k \in S_k \\ 0, \quad \text{if } k \notin S_k. \end{cases} \qquad \ldots(2.1)$$

Then g is the unity of $(F, +, \psi)$.

PROOF : We shall prove that g satisfies (1.1).

Let x_r be the largest element in S_k. By (iv) of Lemma 2.1, $S_k = Sx_r$. Since $x_r \in Sx_r$, from (2.1), it is clear that

$$1 = \sum_{x \in Sx_r} g(x) = \sum_{x \in S_k} g(x). \qquad \ldots(2.2)$$

It remains to prove that if n and k are positive integers with $n < k$, then

$$\sum_{\psi(y, n) = k} g(y) = 0. \qquad \ldots(2.3)$$

We distinguish the following cases (in what follows we tacitly assume the results of Lemma 2.1) :

Case 1—Let $n \in S_n$. Since $g(y) = 0$ if $y \notin S_y$, we may assume that $y \in S_y$ in the sum on the left-hand side of (2.3). Now, $\psi(y, n) = k$ implies that $S_y \subseteq S_k$ and $S_n \subseteq S_k$. Since $y \in S_y$ and $n \in S_n$, $y, n \in S_k$. Hence $k = \psi(y, n) \in S_k$. Thus we have $n \in S_n \subseteq S_k$ and $k \in S_k$. Let $S_k = \{x_1, x_2, \ldots, x_r\}$ with $x_1 < x_2 < \ldots < x_r = k$. Since $n \in S_n \subseteq S_k$, we assume that $n = x_t$ where $x_t \in S_{x_t}$.

First we show that $\sum_{\psi(y,x_i)=x_{i+1}} g(y) = 0$. Since $g(y) = 0$, if $y \notin S_y$, in this sum, we may assume that $y \in S_y$. Now $\psi(y, x_i) = x_{i+1}$ implies that $y \in S_y \subseteq S_{x_{i+1}}$ and $S_{x_i} \subseteq S_{x_{i+1}}$. Since $x_i \in S_{x_i}$, we have $x_i \in S_{x_{i+1}}$; this together with $y \in S_{x_{i+1}}$ implies that $x_{i+1} = \psi(y, x_i) \in S_{x_{i+1}}$.

Let $y \in S_{x_{i+1}}$ and $y \notin S_{x_i}$. Since $x_i \in S x_{i+1}$, $\psi(y, x_i) \in S_{x_{i+1}}$. Also, $\psi(y, x_i) \geqslant x_i$. Hence $\psi(y, x_i) = x_i$ or x_{i+1}, since x_{i+1} is the largest element in $S_{x_{i+1}}$ and no element of $S_{x_{i+1}}$ can exist in the interval (x_i, x_{i+1}) as $S_{x_{i+1}} \subseteq S_k$. Hence $\psi(y, x_i) = x_{i+1}$ since $\psi(y, x_i) = x_i$ implies that $y \in S_{x_i}$. Therefore by (2.2), since $S_{x_i} \subseteq S_{x_{i+1}}$, we have

$$0 = \sum_{\substack{y \in S_{x_{i+1}} \\ y \notin S_{x_i}}} g(y) = \sum_{\psi(y,x_i)=x_{i+1}} g(y).$$

Let us assume that

$$\sum_{\psi(y,x_i)=x_{i+s}} g(y) = 0$$

for all s with $1 \leqslant s < t \leqslant r - i - 1$. We shall prove that

$$\sum_{\psi(y,x_i)=x_{i+t}} g(y) = 0, \quad \text{if } x_{i+t} \in S_{x_{i+t}}.$$

Since $\psi(y, x_i) = x_{i+t}$ implies that $S_{x_i} \subseteq S_{x_{i+t}}$

and $x_i \in S_{x_i}$, we have $x_i \in S_{x_{i+t}}$.

If $y \notin S_{x_i}$, $y \in S_{x_{i+t}}$ then $\psi(y, x_i) \in S_{x_{i+t}}$ and $\psi(y, x_i) > x_i$.

Hence $\psi(y, x_i) = x_{i+1}$ or. Therefore

$$0 = \sum_{\substack{y \in S_{x_{i+t}} \\ y \notin S_{x_i}}} g(y) = \sum_{\substack{y \in S_{x_{i+t}} \\ \psi(y,x_i)=x_{i+1}}} g(y) + \ldots + \sum_{\substack{y \in S_{x_{i+t}} \\ \psi(y,x_i)=x_{i+t}}} g(y). \quad \ldots(2.4)$$

If $x_{i+1} \notin S_{x_{i+t}}$, then $\psi(y, x_i) = x_{i+1}$ does not occur. Hence the first sum on the right-hand side of (2.4) vanishes. Similar remark applies to the other sums on the right-hand side of (2.4). Hence we may assume that $x_{i+s} \in S_{x_{i+t}}$, for $s = 1, 2, \ldots, t$. So, $S_{x_{i+s}} \subseteq S_{x_{i+t}}$, for $s = 1, 2, \ldots, t$. The variable y in each of the sums on the right-hand side of (2.4) can be assumed to be in S_y since $g(y) = 0$ if $y \notin S_y$. This, implies that '$y \in S_{x_{i+t}}$' need not be mentioned in each of the sums on the right-hand side of (2.4). Hence (2.4) can be written as

$$0 = \sum_{\substack{y \in S_{x_{i+t}} \\ y \notin S_{x_i}}} g(y) = \sum_{\psi(y,x_i)=x_{i+1}} g(y) + \ldots + \sum_{\psi(y,x_i)=x_{i+t-1}} g(y) + \sum_{\psi(y,x_i)=x_{i+t}} g(y). \quad \ldots(2.5)$$

By our induction hypothesis, the first $t - 1$ sums on the right side of (2.5) vanish and then we obtain $0 = \sum_{\psi(y,x_i)=x_{i+t}} g(y)$. The induction is complete Therefore,

we have $\sum_{\psi(y,x_i)=k} g(y) = 0$, since $k = x_r$.

Case 2—$n \notin S_n$. Suppose the equation $\psi(y, n) = k$ has a solution y with $y \in S_y$. Then $y \in S_y \subseteq S_k$ and $S_n \subseteq S_k$. Let t be the largest element in S_n so that $t < n$. If $\psi(y, n) = k$, then we also have

$$k = \psi(y, n) = \psi(y, \psi(t, n)) = \psi(\psi(y, t), n)$$

so that $\psi(y, t)$ is also a solution of the equation $\psi(x, n) = k$. Let $Y_1, Y_2, \ldots, Y_r$ be all the elements of S_k which satisfy $\psi(y_i, n) = k$, and $y_i \geqslant t$ for $i = 1, 2, \ldots, r$ and $y_1 < y_2 < \ldots < y_r$. (We may note here that $y_r = x_r$ the largest element in S_k). Also, no $y_i = t$. For if $y_i = t$, then $k = \psi(y_i, n) = \psi(t, n) = n$. But $n < k$. So, $y_i > t$ for $i = 1, 2, \ldots, r$. We have

$$\sum_{\psi(y,n)=k} g(y) = \sum_{\substack{\psi(y,n)=k \\ \psi(y,t)=y_1}} g(y) + \sum_{\substack{\psi(y,n)=k \\ \psi(y,t)=y_2}} g(y) + \ldots + \sum_{\substack{\psi(y,n)=k \\ \psi(y,t)=y_r}} g(y).$$

Now, $\psi(y, t) = y_i$ and $\psi(y_i, n) = k$ imply that

$$k = \psi(y_i, n) = \psi(\psi(y, t), n) = \psi(y, \psi(t, n)) = \psi(y, n)$$

since $t \in S_n$. Hence we have

$$\sum_{\psi(y,n)=k} g(y) = \sum_{\psi(y,t)=y_1} g(y) + \ldots + \sum_{\psi(y,t)=y_r} g(y) \quad \ldots(2.6)$$

since $t < y_i$, $i = 1, 2, \ldots, r$ and $t \in S_t$, each sum on the right-hand side of (2.6) is a sum considered in Case 1 and hence vanishes. Thus

$$\sum_{\psi(y,n)=k} g(y) = 0.$$

The proof of Theorem 2.1 is complete.

We state without proof the following :

Theorem 2.2—Suppose ψ satisfies the postulates (I), (II) and (III) of §1 so that $(F, +, \psi)$ is a commutative ring. If $\psi(x, y) \geqslant \max\ \{x, y\}$ for all $(x, y) \in T$, then

the unity of $(F, +, \psi)$ exists if and only if for each $k \in Z^+$, equation $\psi(x, k) = k$ has a solution. In such a case, the unity is given by (2.1).

Remark 2.1 : If $\psi(x, y) < \max\{x, y\}$ for some $(x, y) \in T$, then the conclusion of Theorem 2.2 need not hold. For example, let $T = \{(1, 2), (2, 1)\} \cup \{(k, k) : k \geqslant 2\}$ and ψ on T be defined by $\psi(1, 2) = \psi(2, 1) = 1$ and $\psi(k, k) = k$ for $k \geqslant 2$. Then ψ satisfies the postulates I, II and III of §1 and clearly for each $k \in Z^+$, $\psi(x, k) = k$ has a solution. Note that $\psi(2, 1) = 1 < 2 = \max\{2, 1\}$. It can be easily shown that (for example using (1.1)) that $(F, +, \psi)$ does not possess the unity.

Remark 2.2 : Let $T = \{(2k, 2k), (2k - 1, 2k), (2k, 2k - 1) ; k \in Z^+\}$. We define $\psi : T \rightarrow Z^+$ by $\psi(x, y) = \min\{x, y\}$ for all $(x, y) \in T$. It can be shown that ψ satisfies the postulates I, II and III of §1 so that $(F, +, \psi)$ is a commutative ring. Also using (1.1) it is not difficult to show that the function of defined by $g(2k) = 1$ and $g(2k - 1) = 0$ for $k = 1, 2, 3, \ldots$, is the unity of $(F, +, \psi)$. Clearly the condition $\psi(x, y) \geqslant \max\{x, y\}$ for all $(x, y) \in T$ is violated.

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ITERATIVE METHODS OF SOLUTIONS FOR LINEAR AND QUASI LINEAR COMPLEMENTARITY PROBLEMS

R. N. MUKHERJEE AND H. L. VERMA

Department of Applied Mathematics, Institute of Techology, Banaras Hindu University, Varanasi 221005

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The present work has been conceived out of a need to extend the method of Ahn to give an iterative procedure for approximating solution of a 'quasi-linear complementarity problem' (QLCP). We also give the bound for spectral radius of them odified matrix in the context of QLCP. Further we give more results on the modification of the algorithm of Pang for finding the solution of QLCP for a pair (M, q), where M is symmetric and positive definite. The fixed parameter approach of Pang also has been modified to incorporate the variable parameter method in successive iteration process.

1. INTRODUCTION

The numerical method for LCP carries with it two methods, e. g., direct and indirect methods. Because of the complexity, the use of direct method is restricted for large size problems. Therefore iterative methods are well suited for such problems. The present work attempts to develop the procedure for finding approximate solutions of Quasi-Linear-Complementarity problems by iterative technique. Essentially this extends the earlier algorithm of Ahn[1] used for solution method for Linear Complementarity problems. Secondly, we also give an extension of a method of Pang[5] to incorporate QLCP for the same purpose. Our attempt next would be to briefly indicate the essential procedure of Pang[5] because we would refer that latter in our extensions.

Consider the symmetric LCP (q, M) :

$$q + Mx \geqslant 0, \quad x \geqslant 0, \text{ and } x^T (q + Mx) = 0$$

where $q \in R^n$ and $M \in R^{n\times n}$ are given and $x \in R^n$. Let (B, C) be a Q-splitting of the matrix M, i. e. $M = B + C$ is a Q-matrix [the LCP (q, B) has a solution for all vectors q]. Let E^k be a non-negative diagonal matrix with $E^k_{ii} < 1$. Define the point to set algorithmic map A^k as follows: for all vectors x,

$$A^k (x^k) = \text{solution set of the LCP } (q + Cx, B, E^k x^k). \qquad \ldots(1.1)$$

The latter LCP (r, B, s) is to find y so that

$$r + By \geqslant 0, \; y \geqslant s \text{ and } (y - s)^T (r + By) = 0.$$

The LCP (r, B, s) can be converted into the LCP $(r + Bs, B)$ if we translate the variable $x = y - s$ since B is a Q-matrix, the set $A^k(x^k)$ is non-empty for all vectors x. Moreover a vector x^* solves the LCP (q, M) if and only if it is fixed point of the map A^k i. e. $x^* \in A^k(x^*)$.

We de fine an iterative technique for solving the LCP (q, M) given the diagonal matrix E^k and the Q-splitting (B, C) of the matrix M. Let $x^0 \geqslant 0$ be an arbitrary non-negative vector. In general $x^k \geqslant 0, k \geqslant 0$ let x^{k+1} be any vector in the set $A^k(x^k)$.

The motivation for using the map $A^k(x^k)$ lies in the fact that the matrix E^k may satisfy the bound $E^k_{ii} < 1$ after the iteration proceeds onwards after a fixed index k_0. This idea is compatible with the usual approach in the contraction mapping case where a fixed power of a mapping may be a contraction although the original map may not be a contraction.

If B is a P-matrix (A real matrix $A \in R^{n \times n}$ is said to be a P-matrix if it has positive principal minors) then the set $A^k(x^k)$ is singleton for all x. In this case, each x^{k+1} will be uniquely defined.

Pang[4] has given necessary and sufficient conditions on the matrix M on the convergence property, i. e. for all vector q and all starting vector $x^\circ \geqslant 0$, each sequence $\{x^k\}$ generated by the iterative technique will converge to some solution of the LCP (q, M).

A Quasi-linear Complementarity (QLCP) can be stated as follows

Find $z \in R^n$ such that

$$z - Qz \geqslant 0, \; Mz + q \geqslant 0 \; (z - Qz)^T (Mz + q) = 0. \qquad \text{...(1.2)}$$

With splitting as indicated previously the point to-set algorithmic map takes the shape.

$$\begin{aligned} A^k(x^k) = \text{Solution set of LCP } (q + BQ\, x^k + C\, x^k, \\ B, E^k (1 - Q) x^k) \end{aligned} \qquad \text{...(1.3)}$$

Variantly x^* solves the QLCP (1.2) if and only if $(1 - Q)\, x^*$ is a point in range of the set valued map A, i. e. a point in $A(x^*)$. The result connected with the convergence of various dual iterative techniques for the solution of strictly convex quadratic program

$$\min_{(1-Q)x \geqslant 0} \; f(x) = q^T x + \tfrac{1}{2} x^T M x \qquad \text{...(1.4)}$$

can be derived by the methods of Pang[5].

We explain some matrix notations as follows : If A is an $n \times m$ matrix, α and β are subsets of $\{1, .., n\}$ and $\{1, .., m\}$, respectively, by $A_{\alpha\beta}$ we denote the sub-matrix of A whose rows and columns are indexed by α and β respectively. If $\alpha = \{1, ..., n\}$,

we denote by A_β the submatrix whose columns of A are indexed by β; similar definition applies to A_α.

2. Preliminaries

We restate again the QLCP as :

find $z \in R^n$, such that

$$z - Qz \geqslant 0,\ Mz + q \geqslant 0, (z - Qz)^T (Mz + q) = 0 \qquad \ldots(2.1)$$

where M is an $n \times n$ real and non-symmetric matrix, q is $n \times 1$ vector. If we take $Q = 0$ in QLCP we get the same LCP as in Ahn[1].

First of all we describe the notations which occur in the QLCP. All matrices and vectors are real. A matrix A with m-rows, n-columns is denoted by $R^{m \times n}$. Row i of matrix A is denoted by A_i and column j by A_j and the element in row i and column j by A_{ij}. The transpose of a matrix is denoted by super script T, such as the transpose of the matrix A is given by A^T, $|A|$ denotes the matrix obtained from the real matrix $A \in R^{m \times n}$ by replacing each element A_{ij} by its absolute value.

If $x \in R^n$, x_+ denotes the vector with elements

$$(x_+)_j = \max \{0, x_j\};\ j = 1, 2, \ldots, n.$$

For any x and y in R^n, it can be easily shown that

(i) $(x + y)_+ \leqslant x_+ + y_+$,

(ii) $x \leqslant y \Rightarrow x_+ \leqslant y_+$.

A real matrix $A \in R^{n \times n}$ is said to be a z-matrix (a P-matrix) if it has non-positive off diagonal entries (positive primal minors).

A square matrix with non-positive cff diagonal elements and wiih a non-negative inverse in called an M-matrix. It can be easily shown that a matrix which is both a Z-matrix and a P-matrix is an M-matrix (or Minkowski matrix).

Given any real matrix $A \in R^{n \times n}$, we define its comparison matrix

$$A_c = (C_{ij})$$

by

$$C_{ii} = |A_{ii}|$$

and

$$C_{ij} = -|A_{ij}|,\ i \neq j,\ i, j = 1, 2, \ldots, n$$

This definition is due to Verga[6]

3. Iterative Algorithm

For solving QLCP (2.1) we describe the general fundamental algorithm.

Lemma 3.1—Let $M \in R^{n \cdot n}$ and E be any positive diagonal matrix, then,

$$z - Qz \geqslant 0,\ Mz + q \geqslant 0,\ (z - Q\,z)^T\,(Mz + q) = 0$$

$$\Leftrightarrow z = \{(1 - Q)\,z - \omega\,E\,(M\,z + q)\}_+,\ \text{for all or some } \omega > 0.$$

Its proof is same as in Mangasarian[2]. This result can be transformed to a fixed point problem for solving the equation $z = f(z)$

where

$$f(z) = \{(1 - Q)z - \omega E\,(Mz + q)\}_+.$$

This result readily leads to the following general algorithm suggested by Mangasarian[2]. We modify this algorithm at certain steps.

Algorithm 3.1—Let $z^0 \geqslant 0$, compute

$$z^{k+1} = \lambda\,[(1 - Q)\,z^k - \omega E^k\,(Mz^k + q + K^k\,(1 - Q)\,(z^{k+1} - z^k))]_+ + (1 - \lambda)\,(1 - Q)\,z^k \qquad \text{...(3.1)}$$

where

$$k = 0, 1, \ldots\ 0 < \lambda \leqslant 1,\ \omega > 0$$

and $\{E^k\}$ and $\{K^k\}$ are bounded sequences of matrices in $R^{n \times n}$, with each E^k being a positive diagonal matrix satisfying $E^k \geqslant \alpha\,I$, for some $\alpha > 0$

where I is the identity matrix.

For the symmetric case Mangasarian has established convergence criteria of this general algorithm. We simplify this algorithm by setting

$$\lambda = 1,\ E^k = E,\ K^k = K,\ \text{for each } k.$$

Remark : As has been indicated in the conclusion, we can relax the above criteria for fixing the matrix powers E^k and K^k as constant matrices to derive certain variable parameter algorithm as well.

Algorithm 3.2—Let $z^0 \geqslant 0$, compute,

$$z^{k+1} = [(1 - Q)\,z^k - \omega E\,(Mz^k + q + K\,(1 - Q)\,(z^{k+1} - z^k))]_+ \qquad k = 0, 1, \ldots \qquad \text{...(3.2)}$$

where $\omega > 0$, E is a positive diagonal matrix and K is either strictly upper triangular or lower triangular matrix. Convergence properties for non-symmetric situations can not be established relying on the descent function of the form

$$\tfrac{1}{2}\,x^T\,M\,x + q^T x$$

so the recurssive relation between two successive iterations will be utilized here.

4. Convergence Properties

First of all we develop the fundamental recursive inequality for Algorithm 3.2 which will be the basis of convergence. This inequality is derived from the inequality properties of x_+ and y_+.

Lemma 4.1—The kth and $(k+1)$ th solutions z^k and z^{k+1} satisfy the partial ordering recurssive inequality :

$$| z^{k+1} - z^k | \leqslant (I - \omega E | K||1 - Q |)^{-1} | (1 - Q)(I + \omega EK) - \omega EM||z^k - z^{k-1} | . \quad \ldots(4.1)$$

From this Lemma we can produce a condition for the sequence $\{z^k\}$ of Algorithm 3.2 to be bounded and have an accumulation point which solves the QLCP (2.1).

If we put $Q = 0$ in (4.1) we have

$$| z^{k+1} - z^k | \leqslant (I - \omega E | K |)^{-1} | I - \omega E (M - K)||z^k - z^{k-1} |$$

which is the standard form given by Ahn[1].

Theorem 4.1—Suppose that the given iteration parameter ω, E, K and the underlying matrix M satisfy

$$\mu ((I - \omega E | K||1 - Q |)^{-1} | (1 - Q)(I + \omega EK) - \omega EM |) < 1 \quad \ldots(4.2)$$

where μ (.) denotes the spectral radius; then the sequence $\{z^k\}$ of Algorithm 3.2 converges to a solution z^* of QLCP.

The proof is similar to Ahn[1].

Here also we can find the same spectral radius which is established by Ahn[1], simply by taking $Q = 0$, in (4.2), viz.,

$$\mu ((I - \omega E | K |)^{-1} | I - \omega E (M - K) |) < 1.$$

We shall start the next section in which we modify the same result on the convergence of iterative methods for the symmetric QLCP. Pang had developed necessary and sufficient condition (for a fixed parameter) for the convergence of iterative method and for solving each individual LCP. We shall extend the method of Pang to incorporate it for the treatment of QLCP.

5. Non-degenerate Case

We classify our analysis into two cases which depends on the nature of the matrix, i. e. either the matrix is non-degenerate or positive semi-definite. Since we know that the matrix M is non-degenerate if all its principal minors are non-zero and the same case for the non-degeneracy in linear complementarity theory[3] the matrix M is non-degenerate if and only if the LCP (q, M) has a finite number of solutions for all vectors q.

The following theorem is the main result of this section.

Theorem 5.1—Let M be symmetric and non-degenerate matrix. Let (B, C) be regular Q-splitting of the matrix M. Let E^k be a non-negative diagonal matrix, with $E^k_{ii} < 1$, for all i. Then the following statements are equivalent:

(A) for some vector q and any initial vector $x^0 \geqslant 0$, any sequence $\{x^k\}$ satisfying $(1 - Q)\, x^{k+1} \in A^k\,(x^k)$ is bounded and thus has at least one accumulation point, moreover, any such point solves the QLCP (q, M).

(B) for some vector q, the quadratic function $f(x) = q^T x + \frac{1}{2} x^T Mx$ is bounded below for $(1 - Q)\, x \geqslant 0$.

(C) for some vector q and any initial vector $x^0 > 0$, any sequence $\{x^k\}$ satisfying $(1 - Q)\, x^{k-1} \in A^k(x^k)$ converges to solution of the QLCP (q, M).

Proof can be given in a line of the arguments given in Pang[5].

6. Concluding Remarks

As one of the concluding remarks we would like to point out that in the case of quasi-linear complementarity problems the algorithm which was developed in section 3, for the iterative solution technique can as well be generalized for variable parameters such as the case when the assumptions $E = E^k$ and $K = K^k$ are relaxed and we take uniformly bounded (by matrix norm) matrices E^k and K^k in the iterative process of the algorithm itself. The variable parameter algorithms are still possible to find the fixed point for the set-valued maps $A\,(x^k)$.

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ON SOME NEW DISCRETE INEQUALITIES IN TWO INDEPENDENT VARIABLES

B. G. Pachpatte

Department of Mathematics and Statistics, Marathwada University Aurangabad 431004, (*Maharashtra*)

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The aim of this paper is to establish some new discrete inequalities in two independent variables which can be used as handy tools in the qualitative analysis of a new class of finite difference equations involving two independent variables.

1. Introduction

The fundamental role played by the discrete inequalities in the development of the theory of finite difference equations and numerical analysis is well known. A large number of papers dealing with discrete inequalities and their applications have appeared during the last few years, see[1-10] and some of the references given therein. Although stimulating research works have been undertaken in this direction, there are still a number of interesting classes of multidimensional finite difference equations which needs new types of discrete inequalities in their analysis. Our objective here is to present some new discrete inequalities in two independent variables which can be used as handy tools in the qualitative analysis of a new class of finite difference equations in two independent variables. In order to convey the importance of our results to the literature, we present applications of some of our inequalities to the study of boundedness, uniqueness and continuous dependence of the solutions of a new class of fourth order finite difference equations in two independent variables.

2. Statement of Results

We first summarise some basic notations and definitions which will be used throughout this paper. Let $N_0 = \{0, 1, 2, \ldots\}$. The expression $u(0) + \sum_{s=0}^{n-1} b(s)$ represents a solution of the linear difference equation $\Delta u(n) = b(n)$ for $n \in N_0$, where Δ is the operator defined by $\Delta u(n) = u(n+1) - u(n)$. The expression $u(0) \prod_{s=0}^{n-1} b(s)$ represents a solution of the linear difference equation $u(n+1) = b(n)\, u(n)$ for $n \in N_0$. We use the usual convention of writing $\sum_{s \in \Phi} b(s) = 0$

and $\prod_{s \in \Phi} b(s) = 1$, if Φ is the empty set. We also use the the following notations of the operators

$$\Delta_1 u(m, n) = u(m+1, n) - u(m, n),$$

$$\Delta_2 u(m, n) = u(m, n+1) - u(m, n)$$

for $m, n \in N_0$. We often use the letters m and n to denote the two independent variables which are members of N_0.

For convenience we list the following hypotheses :

(H_1) $u(m, n)$ and $h(m, n)$ are real-valued nonnegative functions defined for $m, n \in N_0$.

(H_2) $p_1(m, n)$, $p_2(m, n)$, $p_3(m, n)$ are real-valued positive functions defined for $m, n \in N_0$.

(H_3) $a(m, n)$ is real-valued, positive and nondecreasing function in both the variables m and n in N_0.

(H_4) $u(m, n) \geqslant u_0 \geqslant 0$, u_0 is a constant, $h(m, n) > 0$ are real-valued functions defined for $m, n \in N_0$.

(H_5) $g(u)$ is continuous, nondecreasing real-valued function defined on an interval $I = [u_0, \infty)$, $u_0 \geqslant 0$ is a constant, and $g(u) > 0$ on (u_0, ∞), $g(u_0) = 0$.

(H_6) $q_1(m, n)$, $q_2(m, n)$, $q_3(m, n)$ are real-valued positive functions defined for $m, n \in N_0$.

(H_7) $W(u)$ is continuous, nondecreasing and submultiplicative real-valued function defined on an interval I, and $W(u) > 0$ on (u_0, ∞), $W(u_0) = 0$.

A useful two independent variable discrete inequality is embodied in the following theorem.

Theorem 1—Suppose (H_1) and (H_2) are true. If

$$u(m, n) \leqslant c + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \times \sum_{t=0}^{y-1} h(s, t)\, u(s, t) \qquad \ldots(1)$$

for $m, n \in N_0$, where c is a nonnegative constant, then

$$u(m, n) \leqslant c \prod_{x=0}^{m-1} \left[1 + \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \right.$$

$$\times \sum_{t=0}^{y-1} h(s, t) \Big] \qquad \ldots(2)$$

for $m, n \in N_0$.

A slightly different version of Theorem 1 is given in the following theorem.

Theorem 2—Suppose (H_1), (H_2) and (H_3) are true. If

$$u(m, n) \leqslant a(m, n) + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t)\, u(s, t) \qquad \ldots(3)$$

for $m, n \in N_0$, then

$$u(m, n) \leqslant a(m, n) \prod_{x=0}^{m-1} \Big[1 + \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t) \Big] \qquad \ldots(4)$$

for $m, n \in N_0$.

Another interesting and useful discrete inequality is established in the following theorem

Theorem 3—Suppose (H_2), (H_4) and (H_5) are true. If

$$u(m, n) \leqslant c + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t)\, g(u(s, t)) \qquad \ldots(5)$$

for $m, n \in N_0$, where c is a nonnegative constant, then for $0 \leqslant m \leqslant m_1$, $0 \leqslant n \leqslant n_1$, $m, m_1, n, n_1 \in N_0$,

$$u(m, n) \leqslant G^{-1} \Big[G(c) + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t) \Big] \qquad \ldots(6)$$

where

$$G(r) = \int_{r_0}^{r} \frac{dy}{g(y)}, \ r \geqslant u_0 \text{ with } r_0 > u_0 \qquad \ldots(7)$$

G^{-1} is the inverse of G and $m_1, n_1 \in N_0$ are chosen so that

$$G(c) + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \sum_{t=0}^{y-1} \times h(s, t) \in \mathrm{Dom}(G^{-1})$$

for $m, n \in N_0$ and $0 \leqslant m \leqslant m_1$, $0 \leqslant n \leqslant n_1$.

We next establish the following more general inequality which may be convenient in some applications.

Theorem 4—Suppose (H_1), (H_2), (H_6) and (H_7) are true. If

$$\begin{aligned} u(m, n) \leqslant c &+ \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \\ &\times \sum_{t=0}^{y-1} h(s, t)\, u(s, t) \\ &+ \sum_{x=0}^{m-1} \frac{1}{q_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{q_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{q_3(s, y)} \\ &\times \sum_{t=0}^{y-1} k(s, t)\, W(u(s, t)) \end{aligned} \qquad \ldots(8)$$

for $m, n \in N_0$, where c is a nonnegative constant and $k(m, n)$ is a real-valued nonnegative function defined for $m, n \in N_0$, then for $0 \leqslant m \leqslant m_2$, $0 \leqslant n \leqslant n_2$, $m, m_2, n, n_2 \in N_0$

$$\begin{aligned} u(m, n) \leqslant Q(m, n)\, \Omega^{-1} \Bigg[\Omega(c) &+ \sum_{x=0}^{m-1} \frac{1}{q_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{q_2(s, n)} \\ &\times \sum_{y=0}^{n-1} \frac{1}{q_3(s, y)} \sum_{t=0}^{y-1} k(s, t)\, W(Q(s, t)) \Bigg] \end{aligned} \qquad \ldots(9)$$

where

$$Q(m, n) = \prod_{x=0}^{m-1} \Big[1 + \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{p}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t)\Big] \quad \ldots(10)$$

and

$$\Omega(r) = \int_{r_0}^{r} \frac{dy}{W(y)}, \quad r \geqslant u_0 \text{ with } r_0 > u_0 \quad \ldots(11)$$

Ω^{-1} is the inverse of Ω and $m_2, n_2 \in N_0$ are chosen so that

$$\Omega(c) + \sum_{x=0}^{m-1} \frac{1}{q_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{q_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{q_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} k(t, s)\, W(Q(t, s)) \in \text{Dom}(\Omega^{-1})$$

for $m, n \in N_0$ and $0 \leqslant m \leqslant m_2$, $0 \leqslant n \leqslant n_2$.

3. Proofs of Theorems 1—4

In order to establish the inequality (2) in Theorem 1, we first assume that $c > 0$ and define a function $z(m, n)$ by

$$z(m, n) = c + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t)\, u(s, t). \quad \ldots(12)$$

From (12) it is easy to observe that

$$z(0, n) = z(m, 0) = c \quad \ldots(13)$$

and

$$p_1(m, n)\, \Delta_1 z(m, n) = \sum_{s=0}^{m-1} \frac{1}{p_2(s, y)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, n)} \sum_{t=0}^{y-1} h(s, t)\, u(s, t)$$

$$\ldots(14)$$

$$p_2(m,n)\,\Delta_1[p_1(m,n)\,\Delta_1 z(m,n) = \sum_{y=0}^{n-1}\frac{1}{p_3(m,y)}\sum_{t=0}^{y-1} h(m,t)\,u(m,t) \quad \ldots(15)$$

$$p_3(m,n)\,\Delta_2[p_2(m,n)\,\Delta_1[p_1(m,n)\,\Delta_1 z(m,n)]] = \sum_{t=0}^{n-1} h(m,t)\,u(m,t) \quad \ldots(16)$$

$$\Delta_2[p_3(m,n)\,\Delta_2[p_2(m,n)\,\Delta_1[p_1(m,n)\,\Delta_1 z(m,n)]]] = h(m,n)\,u(m,n). \quad \ldots(17)$$

Using the fact that $(u, m, n) \leqslant z(m, n)$ in (17) we have

$$\Delta_2(p_3(m,n)\,\Delta_2[p_2(m,n)\,\Delta_1[p_1(m,n)\,\Delta_1 z(m,n)]]] \leqslant h(m,n)\,z(m,n). \quad \ldots(18)$$

From the definition of $u(m,n)$ we observe that $z(m,n) \leqslant z(m,n+1)$ for $m, n \in N_0$. Using this fact in (18) we see that

$$\frac{p_3(m,n+1)\,\Delta_2[p_2(m,n+1)\,\Delta_1[p_1(m,n+1)\,\Delta_1 z(m,n+1)]]}{z(m,n+1)} - \frac{p_3(m,n)\,\Delta_2[p_2(m,n)\,\Delta_1[p_1(m,n)\,\Delta_1 z(m,n)]]}{z(m,n+1)} \leqslant h(m,n). \quad \ldots(19)$$

From (19) and the fact that $p_3(m,n)\,\Delta_2[p_2(m,n)\,\Delta_1[p_1(m,n)\,\Delta_1 z(m,n)]] \geqslant 0$ from (16), we observe that

$$\frac{p_3(m,n+1)\,\Delta_2[p_2(m,n+1)\,\Delta_1[p_1(m,n+1)\,\Delta_1 z(m,n+1)]]}{z(m,n+1)} - \frac{p_3(m,n)\,\Delta_2[p_2(m,n)\,\Delta_1[p_1(m,n)\,\Delta_1 z(m,n)]]}{z(m,n)} \leqslant h(m,n). \quad \ldots(20)$$

Now keeping m fixed in (20), set $n = t$ and sum over $t = 0, 1, 2, \ldots, n-1$ and use the fact that $p_3(m,0)\,\Delta_2[p_2(m,0)\,\Delta_1[p_1(m,0)\,\Delta_1 z(m,0)]] = 0$, from (16), to obtain the estimate

$$\frac{p_3(m,n)\,\Delta_2[p_2(m,n)\,\Delta_1[p_1(m,n)\,\Delta_1 z(m,n)]]}{z(m,n)} \leqslant \sum_{t=0}^{n-1} h(m,t). \quad \ldots(21)$$

From (21) and in view of the facts that $z(m,n) \leqslant z(m,n+1)$ and $p_2(m,n)\,\Delta_1[p_1(m,n)\,\Delta_1 z(m,n)] \geqslant 0$, we observe that

$$\frac{p_2(m,n+1)\,\Delta_1[p_1(m,n+1)\,\Delta_1 z(m,n+1)]}{z(m,n+1)}$$

$$- \frac{p_2(m, n)\,\Delta_1[p_1(m, n)\,\Delta_1 z(m, n)]}{z(m, n)}$$

$$\leqslant \frac{1}{p_3(m, n)} \sum_{t=0}^{n-1} h(m, t). \qquad \ldots(22)$$

Keeping m fixed in (22) set $n = y$ and sum over $y = 0, 1, 2, \ldots, n-1$ and use the fact that $p_2(m, 0)\,\Delta_1[p_1(m, 0)\,\Delta_1 z(m, 0)] = 0$ from (15), to obtain the estimate

$$\frac{p_2(m, n)\,\Delta_1[p_1(m, n)\,\Delta_1 z(m, n)}{z(m, n)} \leqslant \sum_{y=0}^{n-1} \frac{1}{p_3(m, y)} \sum_{t=0}^{y-1} h(m, t). \qquad \ldots(23)$$

From (23) and in view of the facts that $z(m, n) \leqslant z(m+1, n)$ and $p_1(m, n)\,\Delta_1 z(m, n) \geqslant 0$ from (14), we observe that

$$\frac{p_1(m+1, n)\,\Delta_1 z(m+1, n)}{z(m+1, n)} - \frac{p_1(m, n)\,\Delta_1 z(m, n)}{z(m, n)}$$

$$\leqslant \frac{1}{p_2(m, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(m, y)} \sum_{t=0}^{y-1} h(m, t). \qquad \ldots(24)$$

Now keeping n fixed in (24), set $m = s$ and sum over $s = 0, 1, 2, \ldots, m-1$ and use the fact that $p_1(0, n)\,\Delta_1 z(0, n) = 0$ from (14), to obtain the estimate

$$\frac{\Delta_1 z(m, n)}{z(m, n)} \leqslant \frac{1}{p_1(m, n)} \sum_{s=0}^{m-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \sum_{t=0}^{y-1} h(s, t). \qquad \ldots(25)$$

From (25) we see that

$$z(m+1, n) \leqslant z(m, n)\Bigg[1 + \frac{1}{p_1(m, n)} \sum_{s=0}^{m-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t)\Bigg]. \qquad \ldots(26)$$

Now keeping n fixed in (26), set $n = x$ and substitute $x = 0, 1, 2, \ldots, m-1$ successively and use the fact that $z(0, n) = c$ from (13), to obtain the estimate

$$z(m, n) \leqslant c \prod_{x=0}^{m-1} \Bigg[1 + \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t) \Big].$$

Substituting this bound on $z(m, n)$ on the right side of (1) we obtain the inequality in (2).

Now suppose $c = 0$. Then from (1) we see that the inequality

$$u(m, n) \leqslant \epsilon + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t)\, u(s, t)$$

holds for every arbitrary positive number ϵ and $m, n \in N_0$, which by the above argument yields the estimate

$$u(m, n) \leqslant \epsilon \prod_{x=0}^{m-1} \Big[1 + \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t) \Big]. \qquad \ldots(27)$$

Since $u(m, n) \geqslant 0$ and $\epsilon > 0$ is arbitrary number independent of m, n then from (27) it follows that $u(m, n) = 0$. This completes the proof of Theorem 1.

Since $a(m, n)$ is positive and nondecreasing, we observe from (3) that

$$\frac{u(m, n)}{a(m, n)} \leqslant 1 + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\sum_{t=0}^{y-1} h(s, t) \frac{u(s, t)}{a(s, t)}.$$

Now an application of Theorem 1 yields the required bound in (4) and the proof of Theorem 2 is complete.

In order to establish the inequality (6) in Theorem 3, let $\epsilon > 0$ and $u_\epsilon(m, n) = u(m, n) + \epsilon \geqslant u_0$ for all $m, n \in N_0$. Then from (5) we see that

$$u_\epsilon(m, n) \leqslant c + \epsilon + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t)\, g(u_\epsilon(s, t) - \epsilon)$$

$$\leqslant c + \epsilon + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t)\, g(u_\epsilon(s, t)). \qquad \text{..(28)}$$

Define a function $z(m, n)$ by

$$z(m, n) = c + \epsilon + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t)\, g(u_\epsilon(s, t)). \qquad \text{...(29)}$$

From (29) it is easy to observe that

$$z(m, 0) = z(0, n) = c + \epsilon \qquad \text{...(30)}$$

and

$$p_1(m, n)\, \Delta_1\, z(m, n) = \sum_{s=0}^{m-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \sum_{t=0}^{y-1} h(s, t)\, g(u_\epsilon(s, t)). \qquad \text{...(31)}$$

$$p_2(m, n)\, \Delta_1 [p_1(m, n)\, \Delta_1\, z(m, n)] = \sum_{y=0}^{n-1} \frac{1}{p_3(m, y)} \sum_{t=0}^{y-1} h(m, t)\, g(u_\epsilon(m, t)) \qquad \text{...(32)}$$

$$p_3(m, n)\, \Delta_2 [p_2(m, n)\, \Delta_1 [p_1(m, n)\, \Delta_1\, z(m, n)]] = \sum_{t=0}^{n-1} h(m, t)\, g(u_\epsilon(m, t)) \qquad \text{...(33)}$$

$$\Delta_2 [p_3(m, n)\, \Delta_2 [p_2(m, n)\, \Delta_1 [p_1(m, n)\, \Delta_1\, z(m, n)]]] = h(m, n)\, g(u_\epsilon(m, n)). \qquad \text{...(34)}$$

Using the fact that $u_\epsilon(m, n) \leqslant z(m, n)$ in (34) we have

$$\Delta_2 [p_3(m, n) \Delta_2 [p_2(m, n) \Delta_1 [p_1(m, n) \Delta_1 z(m, n)]]] \leqslant h(m, n) g(z(m, n)). \quad \ldots(35)$$

From the definition of $z(m, n)$ in (29) we observe that $z(m, n) \leqslant z(m, n+1)$ for $m, n \in N_0$. Using this and the fact that

$$p_3(m, n) \Delta_2 [p_2(m, n) \Delta_1 [p_1(m, n) \Delta_1 z(m, n)]] \geqslant 0$$

from (33), we observe from (35) that

$$\frac{p_3(m, n+1) \Delta_2 [p_2(m, n+1) \Delta_1 [p_1(m, n+1) \Delta_1 z(m, n+1)]]}{g(z(m, n+1))} - \frac{p_3(m, n) \Delta_2 [p_2(m, n) \Delta_1 [p_1(m, n) \Delta_1 z(m, n)]]}{g(z(m, n))} \leqslant h(m, n). \quad \ldots(36)$$

Now by following exactly the same steps as in the proof of Theorem 1 below the inequality (20) up to the inequality (25) with suitable changes, we obtain

$$\frac{\Delta_1 z(m, n)}{g(z(m, n))} \leqslant \frac{1}{p_1(m, n)} \sum_{s=0}^{m-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \sum_{t=0}^{y-1} h(s, t). \quad \ldots(37)$$

From (7) and (37) we have

$$G(z(m+1, n)) - G(z(m, n)) = \int_{z(m,n)}^{z(m+1,n)} \frac{dy}{g(y)} \leqslant \frac{\Delta_1 z(m, n)}{g(z(m, n))}$$

$$\leqslant \frac{1}{p_1(m, n)} \sum_{s=0}^{m-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \times \sum_{t=0}^{y-1} h(s, t). \quad \ldots(38)$$

Now keeping n fixed in (38), set $m = x$ and sum over $x = 0, 1, 2, \ldots, m-1$ to obtain the estimate

$$G(z(m, n)) \leqslant G(c+\epsilon) + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{n-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \times \sum_{t=0}^{y-1} h(s, t). \quad \ldots(39)$$

The bound in (6) now follows by substituting the bound for $z(m, n)$ from (39) in (28) and letting $\epsilon \to 0$. The subintervals of N_0 for m, n are obvious and the proof of Theorem 3 is complete.

In order to prove the inequality (9) in Theorem 4, let $\epsilon > 0$ and $u_\epsilon(m, n) = u(m, n) + \epsilon \geqslant u_0$ for $m, n \in N_0$. Then from (8) we see that

$$u_\epsilon(m, n) \leqslant c + \epsilon + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t)(u_\epsilon(s, t) - \epsilon) + \sum_{x=0}^{m-1} \frac{1}{q_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{q_2(s, n)}$$

$$\times \sum_{y=0}^{n-1} \frac{1}{q_3(s, n)} \sum_{t=0}^{y-1} k(s, t) W(u_\epsilon(s, t) - \epsilon)$$

$$\leqslant c + \epsilon + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t) u_\epsilon(s, t) + \sum_{x=0}^{m-1} \frac{1}{q_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{q_2(s, n)}$$

$$\times \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \sum_{t=0}^{y-1} k(s, t) W(u_\epsilon(s, t)). \qquad \ldots(40)$$

Define

$$a(m, n) = c + \epsilon + \sum_{x=0}^{m-1} \frac{1}{q_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{q_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{q_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} k(s, t) W(u_\epsilon(s, t)) \qquad \ldots(41)$$

then (40) can be restated as

$$u_\epsilon(m, n) \leqslant a(m, n) + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t) u_\epsilon(s, t).$$

Since $a(m, n)$ is positive and nondecreasing function in both the variables m and n, we have from Theorem 2

$$u_\varepsilon(m, n) \leqslant a(m, n)\, Q(m, n) \tag{42}$$

where $Q(m, n)$ is as defined in (10). Since W is submultiplicative, we have

$$W(u_\varepsilon(m, n)) \leqslant W(a(m, n))\, W(Q(m, n)). \tag{43}$$

From (41) and (43) we have

$$a(m, n) \leqslant c + \epsilon + \sum_{x=0}^{m-1} \frac{1}{q_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{q_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{q_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} k(s, t)\, W(Q(s, t))\, W(a(s, t)).$$

Now by following the proof of Theorem 3 with suitable modifications we obtain

$$a(m, n) \leqslant \Omega^{-1}\Big[\Omega(c + \epsilon) + \sum_{x=0}^{m-1} \frac{1}{q_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{q_2(s, n)}$$

$$\times \sum_{y=0}^{n-1} \frac{1}{q_3(s, y)} \sum_{t=0}^{y-1} k(s, t)\, W(Q(s, t))\Big]. \tag{44}$$

The desired bound in (9) now follows by substituting (44) in (42) and letting $\epsilon \to 0$. The subintervals of N_0 for m and n are obvious. This completes the proof of Theorem 4.

4. Some Applications

In this section, we present some applications of our results to the study of boundedness, uniqueness and continuous dependence of the solutions of a new class of nonlinear finite difference equations in two independent variables. Each of these applications could be stated formally as a theorem. This has not been done so as not to obscure the essential ideas with technical details.

Example 1—As a first application, we obtain a bound on the solution of a nonlinear fourth order finite difference equation

$$\Delta_2 [a_3(m, n)\, \Delta_2 [a_2(m, n)\, \Delta_1 [a_1(m, n)\; \Delta_1 u(m, n)]]] = f(m, n, u)(m, n)) \tag{45}$$

with the given boundary conditions at $m = 0$, $n = 0$

$$u(0, n) = \phi_1(n)$$

$$a_1(0, n)\, \Delta_1 u(0, n) = \phi_2(n)$$

$$a_2(m,0)\ \Delta_1[a_1(m,0)\,\Delta_1 u(m,0)] = \psi_1(m)$$

$$a_3(m,0)\ \Delta_2[a_2(m,0)\,\Delta_1[a_1(m,0)\ \Delta_1 u(m,0)]] = \psi_2(m). \qquad ...(46)$$

Here a_1, a_2, a_3 are real valued positive functions defined on N_0^2, $f: N_0^2 \times R \to R$, where R denotes the set of real numbers; $\phi_1(n)$, $\phi_2(n)$, $\psi_1(m)$, $\psi_2(m)$ are real-valued nonnegative functions defined for $m, n \in N_0$. We assume that

$$|f(m,n,u)| \leqslant h(m,n)\,|u| \qquad ...(47)$$

where $h(m,n)$ is a real-valued nonnegative function defined for $m, n \in N_0$. It is easy to observe that the problem (45) – (46) is equivalent to the equation

$$u(m,n) = b(m,n) + \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s,y)}$$

$$\times \sum_{t=0}^{y-1} f(s,t,u(s,t)) \qquad ...(48)$$

where

$$b(m,n) = \phi_1(n) + \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \phi_2(n) + \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)}$$

$$\sum_{s=0}^{x-1} \frac{1}{a_2(s,n)} \psi_1(s) + \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{a_2(x,n)}$$

$$\times\ \psi_2(s) \sum_{y=0}^{n-1} \frac{1}{a_3(s,y)}. \qquad ...(49)$$

Suppose that

$$|b(m,n)| \leqslant k \qquad ...(50)$$

where k is a nonnegative constant. Using (47), (50) in (48) we have

$$|u(m,n)| \leqslant k + \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s,y)}$$

$$\times \sum_{t=0}^{y-1} h(s,t)\,|u(s,t)|.$$

Now an application of Theorem 1 yields the bound on the solution $u(m,n)$ of (45)-(46) in terms of the known functions.

Example 2—As a second application, we shall discuss the uniqueness of the solution of the problem (45)–(46). We assume that the function f in (45) satisfies

$$|f(m, n, u) - f(m, n, \bar{u})| \leqslant h(m, n) \, |u - u| \qquad ...(51)$$

where $h(m, n)$ is as in Example 1. The problem (45)—(46) is equivalent to the equation (48). Then for any two solutions u and $\bar{u}$ of (45)—(46) we have

$$|u(m, n) - \bar{u}(m, n)| \leqslant \epsilon + \sum_{x=0}^{m-1} \frac{1}{a_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s, n)}$$

$$\times \sum_{y=0}^{n-1} \frac{1}{a_3(s, y)} \sum_{t=0}^{y-1} h(s, t) \, |u(s, t) - \bar{u}(s, t)| \qquad ...(52)$$

where $\epsilon > 0$ is arbitrary constant. The assumption (51) is used to get the inequality in (52). Now an application of Theorem 1 yields

$$|u(m, n) - \bar{u}(m, n)|$$

$$\leqslant \epsilon \left\{ \prod_{x=0}^{n-1} \left[1 + \frac{1}{a_1(x, n)} \sum_{s=u}^{x-1} \frac{1}{a_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s, y)} \sum_{t=0}^{y-1} h(s, t) \right] \right\}.$$

Since $\epsilon > 0$ is arbitrary we have $u = \bar{u}$ i. e. there is at most one solution of the problem (45) – (46).

Example 3—Our third application is an example of continuous dependence of the solution on the equation and boundary data. Consider the problem (45)—(46) in Example 1 and the problem

$$\Delta_2 [a_3(m, n) \Delta_2 [a_2(m, n) \Delta_1 [a_1(m, n) \Delta_1 z(m,n)]]] = F(m, n, z(m, n)) \qquad ...(53)$$

with the given boundary conditions at $m = 0, n = 0$

$$z(0, n) = \bar{\phi}_2(n)$$

$$a_1(0, n) \Delta_1 z(0, n) = \bar{\psi}_2(n)$$

$$a_2(m, 0) \Delta_1 [a_1(m, 0) \Delta_1 z(m, 0)]] = \bar{\psi}_1(m)$$

$$a_3(m, 0) \Delta_2 [a_2(m, 0) \Delta_1 [a_1(m, 0) \Delta_1 z(m, 0)]] = \bar{\psi}_2(m). \qquad ...(54)$$

Here a_1, a_2, a_3 are as in Example 1, $F : N_0^2 \times R \rightarrow R$, $\bar{\phi}_1(n)$, $\bar{\phi}_2(n)$, $\bar{\psi}_1(m)$, $\bar{\psi}_2(m)$ are real-valued nonnegative functions defined for $m, n \in N_0$. The equations equivalent to (45) – (46) and (53) – (54) are (48) and

$$z(m, n) = \bar{b}(m, n) + \sum_{x=0}^{m-1} \frac{1}{a_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} F(s, t, z(s, t)) \qquad \ldots(55)$$

where $\bar{b}(m, n)$ is obtained from the definition of $b(m, n)$ by replacing $\phi_1(n)$, $\phi_2(n)$, $\psi_1(m)$, $\psi_2(m)$ in the right side in (49) by $\bar{\phi}_1(n)$, $\bar{\phi}_2(n)$, $\bar{\psi}_1(m)$, $\bar{\psi}_2(m)$ respectively. From (48) and (55) we have

$$u(m, n) - z(m, n)$$

$$= b(m, n) - \bar{b}(m, n)$$

$$+ \sum_{x=0}^{m-1} \frac{1}{a_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} \{f(s, t, u(s, t)) - F(s, t, z(s, t)\}. \qquad \ldots(56)$$

Suppose that the function f in (45) satisfies the condition (51) and further we assume that

$$|b(m, n) - \bar{b}(m, n)| \leqslant \epsilon \qquad \ldots(57)$$

$$\sum_{x=0}^{m-1} \frac{1}{a_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s, n)} \sum_{y=0}^{x-1} \frac{1}{a_3(s, y)} \sum_{t=0}^{y-1} |f(s, t, z(s, t))$$

$$- F(s, t, z)(s, t))| \leqslant \epsilon \qquad \ldots(58)$$

where $\epsilon > 0$ is arbitrary constant. By substracting and adding $f(s, t, z(s, t))$ in the braces on the right side of equation (56) and using (51), (57), (58), we obtain

$$|u(m, n) - z(m, n)| \leqslant 2\epsilon + \sum_{x=0}^{m-1} \frac{1}{a_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s, n)}$$

$$\sum_{y=0}^{n-1} \frac{1}{a_3(s, y)} \sum_{t=0}^{y-1} h(s, t)\,|u(s, t) - z(s, t)|. \qquad \ldots(59)$$

Now an application of Theorem 1 yields

$$|u(m, n) - z(m, n)|$$

$$\leqslant 2\epsilon \left\{ \prod_{x=0}^{m-1} \left[1 + \frac{1}{a_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s, y)} \right.\right.$$

$$\left.\left. \sum_{t=0}^{y-1} h(s, t) \right] \right\} . \qquad \ldots(60)$$

If $h(m, n)$ is bounded on some compact set $0 \leqslant m \leqslant m_0$, $0 \leqslant n \leqslant n_0$, m, m_0, $n, n_0 \in N_0$, then the quantity in braces on the right in (60) is bounded by some constant M on the set $0 \leqslant m \leqslant m_0$, $0 \leqslant n \leqslant n_0$. Therefore $| u(m, n) - z(m, n) | \leqslant 2M\epsilon$ on the set $0 \leqslant m \leqslant m_0$, $0 \leqslant n \leqslant n_0$; so the solution $u(m, n)$ of (45) – (46) depends continuously on f and the boundary data. If $\epsilon \to 0$, then $| u(m, n) - z(m, n) | \to 0$ on this set.

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PERIODIC BOUNDARY VALUE PROBLEMS FOR AN INFINITE SYSTEM OF NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

K. NARSIMHA REDDY

Department of Mathematics, P. G. College, Osmania University, Secunderabad 500003

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Results on existence of solutions to Periodic Boundary Value Problems for an infinite System of nonlinear second order differential equationshave been discussed and a uniqueness result is presented.

1. INTRODUCTION

There is a large literature on the existence of solutions for periodic boundary value problems (PBVP's for short) of nonlinear scalar differential equations[7-10,12,14]. Results on the existence of solutions for first and second order PBVP's have been obtained earlier[7,8,10,14] by combining two basic techniques, namely the method of upper and lower solutions and the alternative method. In deed, a number of existence theorems for periodic solutions have been obtained by the technique of upper and lower solutions, and others have been and are being obtained by the alternative method. Kannan and Lakshmikantham[6,7] showed that the use of a result proved by Cesari and Kannan[3] by the alternative method could remarkably improve the arguments by upper and lower solutions. New existence theorems could be obtained and previous ones could be obtained by a more uniform approach. Further work in this direction was done by Rao and Vatsala[15]. For an exposition on the alternative method, though preceeding the work in Cesari and Kannan[3], we refer to Cesari[2]. Also we refer the readers to Bernfeld and Lakshnikantham[1] for a general treatment of boundary value problems. Extremal solutions for nonlinear boundary value problems have been obtained by many workers[4,5,8,9,11,16,17] by employing monotone iterative scheme and this method is found to be constructive. PBVP's for infinite systems of first and second order equations have been studied earlier[8,11,14,15,17] and these results extend the results obtained for scalar equations.

In this paper we consider, as it was done earlier [4,5-7,16], an infinite system of second order ordinary differential equations with periodic boundary conditions and discuss the questions of existence and uniqueness of solutions again by combining the method of upper and lower solutions with the result in Cesari and Kannan[3] from the alternative method. The organization of the present paper is as follows : Our notations and terminology are fairly consistent and can be understood by referring to earlier workers[5,11,15,16]. However for the sake of completeness we describe them briefly in section 2. Section 3 deals with existence results employing the alternative method. In

section 4, we develop a monotone iterative technique to obtain coupled extremal quasi-solutions for these systems. The uniqueness results are discussed in section 5.

2. Notations and Terminology

Let I be the interval $[0, 2\pi]$ and $E = R^\infty = R \times R \times \ldots$ in which $R = (-\infty, \infty)$. Also let Z^+ denote the set of all positive integers.

We consider the PBVP

$$-u_i'' = f_i(t, u, u_i'), \; t \in I, i \in Z^+ \qquad \ldots(2.1)$$

$$u(0) = u(2\pi), u'(0) = u'(2\pi)$$

where $f : I \times E \times R \to E$ is continuous.

For each $i \in Z^+$, we define two sets P_i and Q_i such that $Z^+ - \{i\} \quad P_i \cup Q_i$. Hereafter $P(q)$ denotes the generic element of $P_i(Q_i)$ respectively whenever the set is nonempty. Moreover the vector $u \in E$ may be written as $u = (u_i, [u]_p, [u]_q)$, for $p \in P_i$, $q \in Q_i$. Then the PBVP (2.1) becomes

$$\left.\begin{aligned} -u_i'' &= f_i(t, u_i, [u]_p, [u]_q, u_i') \\ u(0) &= u(2\pi), u'(0) = u'(2\pi). \end{aligned}\right\} \qquad \ldots(2.2)$$

Without further mention we assume that $i \in Z^+$ and all the inequalities between vectors hold component wise.

We state the following assumptions which will be used in our subsequent discussion :

(A_0) $\alpha, \beta \in C^2[I, E]$, $\alpha(t) \leqslant \beta(t)$, $t \in I$

(A_1) (i) $\alpha(0) = \alpha(2\pi)$, $\alpha'(0) \geqslant \alpha'(2\pi)$ and $-\alpha_i'' \leqslant f_i(t, \sigma, \alpha_i')$

for all σ such that $\alpha(t) \leqslant \sigma \leqslant \beta(t)$ and $\sigma_i = \alpha_i(t)$, $t \in (0, 2\pi]$

(ii) $\beta(0) = \beta(2\pi)$, $\beta'(0) \leqslant \beta'(2\pi)$ and $-\beta_i'' \geqslant f_i(t, \sigma, \beta_i')$

for all σ such that $\alpha(t) \leqslant \sigma \leqslant \beta(t)$ and $\sigma_i = \beta_i(t)$, $t \in (0, 2\pi]$

(A_2) For $t \in I$, $\alpha(t) \leqslant u(t) \leqslant \beta(t)$ and $v_i \in R$ we have

$$|f_i(t, u, v_i)| \leqslant \begin{cases} h_i(|v_i|) \text{ if } |v_i| \leqslant d_i \\ h_i(d_i) \text{ if } |v_i| > d_i \end{cases}$$

for some $d_i > e_i = 1/2\pi \max \{\alpha_i(0) - \beta_i(2\pi)|, |\alpha_i(2\pi) - \beta_i(0)|\}$, $h_i : [0, \infty) \to (0, \infty)$ is continuous for each i.

Also there exists $N > 0$ depending only on α, β and h such that

$$\int_{e_i}^{l_i} \frac{s\,ds}{h_i(s)} > \max_I \beta_i(t) - \min_I \alpha_i(t).$$

where

$$l_i = \min\{d_i, N_i\}$$

(A_3) f is completely continuous on $I \times E \times R$.

(A_4) For $t \in I$, $\alpha(t) \leqslant v \leqslant u \leqslant \beta(t)$ and $|z_i| \leqslant \bar{d}_i$

$$f_i(t, u_i, [u]_p, [u]_q, z_i) - f_i(t, v_i, [u]_p, [u]_q, z_i) \geqslant -M_i(u_i - v_i)$$

for some $M_i > 0$. Where $\bar{d}_i = \max\left\{N_i, \max_I |\alpha_i'(t)|, \max_I |\beta_i'(t)|\right\}$

(A_5) f possess a mixed quasi-monotone property $(mq\ mp)$ that is $f_i(t, u_i, [u]_p, [u]_q, z_i)$ is monotone nondecreasing in $[u]_p$ and monotone nonincreasing in $[u]_q$.

The functions $\alpha, \beta \in C^2[I, E]$ with $\alpha(t) \leqslant \beta(t)$ on I are said to be coupled lower and upper quasi-solutions of (2.1) respectively if

$$-\alpha_i'' \leqslant f_i\left(t, \alpha_i, [\alpha]_p, [\beta]_q, \alpha_i'\right), \alpha(0) = \alpha(2\pi), \alpha'(0) \geqslant \alpha'(2\pi)$$

$$-\beta_i'' \geqslant f_i\left(t, \beta_i, [\beta]_p, [\alpha]_q, \beta_i'\right), \beta(0) = \beta(2\pi), \beta'(0) \leqslant \beta'(2\pi).$$

The functions $x, y \in C^2[I, E]$ are said to be coupled quasi-solutions of (2.1) if

$$-x_i'' = f_i\left(t, x_i, [x]_p, [y]_q, x_i'\right), x(0) = x(2\pi), x'(0) = x'(2\pi)$$

$$-y_i'' = f_i\left(t, y_i, [y]_p, [x]_q, y_i'\right), y(0) = y(2\pi), y'(0) = y'(2\pi).$$

In the special case where all Q_i's are empty, quasi-solutions are just solutions and in the case P_i's are empty, the quasi-solutions that result are most useful since they may be obtained most easily. We can also define coupled minimal and maximal quasi-solutions analogously.

We state the lemma which is a modified version of a known result.

Lemma 2.1—Let the assumptions (A_0) and (A_2) hold, then for any solution $u \in C^2[I, E]$ of (2.1) with $\alpha(t) \leqslant u(t) \leqslant \beta(t)$ on I, we have

$$|u'(t)| \leqslant N \text{ on } I.$$

The proof of this lemma follows from Lemma 1.1 of Das and Devasahayam.

3. Existence Results

In this section we discuss the existence of solutions of PBVP's by the device, already used many times in the method of upper and lower solutions, of defining the function F as follows :

$$F_i(t, u, v_i) = f_i(t, p(t, u), v_i) + r_i(t, u)$$

where

$$P_i(t, u) = \max\{\alpha_i(t), \min(u_i, \beta_i(t))\}$$

and

$$r_i(t, u) = \begin{cases} \dfrac{\beta_i(t) - u_i}{1 + u_i^2}, & \text{if } u_i > \beta_i(t) \\ 0, & \text{if } \alpha_i(t) \leqslant u_i \leqslant \beta_i(t) \\ \dfrac{\alpha_i(t) - u_i}{1 + u_i^2}, & \text{if } u_i < \alpha_i(t). \end{cases}$$

Consider the following PBVP

$$-u_i'' = F_i\left(t, u, u_i'\right), u(0) = u(2\pi), u'(0) = u'(2\pi). \qquad \ldots(3.1)$$

Lemma 3.1—Let (A_0) and (A_1) hold and let u be a solution of (3.1). Then

$\alpha(t) \leqslant u(t) \leqslant \beta(t)$ on I.

Proof : First we claim that $\alpha(t) \leqslant u(t)$ on I. Suppose not, then we can find a $t_0 \alpha I$ and an $\epsilon > 0$ such that for some $k \in Z^+$

$$\alpha_k(t_0) = u_k(t_0) + \epsilon, \ \alpha_i(t) \leqslant u_i(t) + \epsilon \ \ t \in I. \qquad \ldots(3.2)$$

If $t_0 \in (0, 2\pi)$, we have $\alpha_k'(t_0) = u_k'(t_0)$ and $\alpha_k''(t_0) \leqslant u_k''(t_0)$. From (3.2) $\alpha_k(t_0) > u_k(t_0)$ and hence $p_k(t_0, u(t_0)) = \alpha_k(t_0)$. In view of (A_1) (i) and using the definition of F we have

$$\begin{aligned} f_k(t_0 \sigma, \alpha_k'(t_0)) &\geqslant -\alpha_k''(t_0) \\ &\geqslant -u_k''(t_0) = F_k(t_0, u(t_0), u_k'(t_0)) \\ &> f_k(t_0, p(t_0 u(t_0)), \alpha_k'(t_0)). \end{aligned}$$

Since $\alpha(t) \leqslant p(t, u(t)) \leqslant \beta(t)$ and $p_k(t_0, u(t_0)) = \alpha_k(t_0)$, we get a contradiction by choosing $\sigma = p(t_0, u(t_0))$.

If $t_0 = 0$, from (3.2) we obtain

$$\alpha_k(0) = u_k(0) + \epsilon = \alpha_k(2\pi) \text{ and } \alpha_k'(0) \leqslant u_k'(0) \text{ and } \alpha_k'(2\pi) \geqslant u_k'(2\pi),$$
since $\alpha_k(0) = \alpha_k(2\pi)$, $u_k(0) = u_k(2\pi)$.

And in view of (A_1) (i) it follows that $\alpha_k'(2\pi) = u_k'(2\pi)$ and as before

$$f_k(2\pi, \sigma, \alpha_k'(2\pi)) = -\alpha_k''(2\pi) = -u_k''(2\pi) = F_k(2\pi, u(2\pi), u_k'(2\pi))$$

$$> f_k(2\pi, p(2\pi, u(2\pi)), \alpha_k'(2\pi)).$$

Since $\alpha(2\pi) = p(2\pi, u(2\pi)) \leqslant \beta(2\pi)$ and $p_k(2\pi, u(2\pi)) = \alpha_k(2\pi)$, we again get a contradiction by choosing $\sigma = p(2\pi, u(2\pi))$. On similar lines we can prove that $u(t) \leqslant \beta(t)$ on I and this completes the proof.

Lemma 3.2—Let the assumptions $(A_0) - (A_2)$ hold. Then there exist α_0, β_0 such that the following are true :

$\left(A_0^*\right)$ $\alpha_0, \beta_0 \in C^2[I, E]$, $\alpha_0(t) < \beta_0(t)$, $t \in I$

$\left(A_1^*\right)$ (i) $\alpha_0(0) = \alpha_0(2\pi)$, $\alpha_0'(0) \geqslant \alpha_0'(2\pi)$ and

$$-\alpha_{0,i}'' < F_i\left(t, \bar{\sigma}\ \alpha_{0,i}'\right) \text{ for } \bar{\sigma} \text{ such that}$$

$$\alpha_0(t) \leqslant \bar{\sigma} \leqslant \beta_0(t) \text{ and } \bar{\sigma}_i = \alpha_{0,i}(t),\ t \in (0, 2\pi].$$

(ii) $\beta_0(0) = \beta_0(2\pi)$, $\beta_0'(0) \leqslant \beta_0'(2\pi)$ and

$$-\beta_{0,i}'' > F_i\left(t, \bar{\sigma},\ \beta_{0,i}'\right) \text{ for all } \bar{\sigma} \text{ such that}$$

$$\alpha_0(t) \leqslant \bar{\sigma} \leqslant \beta_0(t) \text{ and } \bar{\sigma}_i = \beta_{0,i}(t),\ t \in (0, 2\pi].$$

$\left(A_2^*\right)$ The condition (A_2) holds with F replacing f and with respect to the pair (α_0, β_0).

PROOF : Let $a_i > 0$, $b_i > 0$ for all $i \in Z^+$ be numbers and define $\alpha_{0,i}(t) = \alpha_i(t) - a_i$ and $\beta_{0,i}(t) = \beta_i(t) + b_i$. Then it is easy to see following the proof of Lemma 2.2 Rao and Vatsala[15] that $\left(A_0^*\right)$ and $\left(A_1^*\right)$ hold. However for the sake of completeness we establish $\left(A_2^*\right)$.

For $t \in I$, $\alpha(t) \leqslant u(t) \leqslant \beta(t)$ and $u_i' \in R$, we have

$$| F_i(t, u, u_i') | = | f_i(t, u, u_i') |$$

$$= \begin{cases} h_i(| u_i' |), & \text{if } | u_i' | \leqslant d_i \\ h_i(d_i), \text{ if } | u_i' | > d_i. \end{cases}$$

Since $h_i(s)$ is a positive constant for $s \geqslant d_i$, there exists an $N_i^* > d_i$ such that

$$\int_{e_i^*}^{N_i^*} \frac{s\,d\,s}{h_i(s)} \int_{e_i^*}^{N_i^*} \frac{sds}{h_i(s)} > \max_I \beta_{0,i}(t) - \min_I \alpha_{0,i}(t)$$

where

$$e_i^* = \min\{e_i, \bar{e}_i\} \text{ and } \bar{e}_i = \frac{1}{2\pi} \max[\,|\alpha_{0,i}(0) - \beta_{0,i}(2\pi)|\,,$$

$$|\alpha_{0,i}(2\pi) - \beta_{0,i}(0)|\,].$$

This proves that (A_2) holds with F_i replacing f_i, $i \in Z^+$.

For $\sigma, \bar{\sigma}$ such that $\alpha_0(t) \leqslant \sigma, \bar{\sigma} \leqslant \beta_0(t)$, $\sigma_i = \alpha_{0,i}$ and $\bar{\sigma}_i = \beta_{0,i}$, define

$$G_i(t, u) = \begin{cases} F_i(t, \bar{\sigma}, \beta_{0,i}') + \dfrac{\beta_{0,i} - u_i}{1 + u_i^2} \text{ if } u_i > \beta_{0,i} \\[2ex] \dfrac{u_i - \alpha_{0,i}}{\beta_{0,i} - \alpha_{0,i}}\left[F_i(t, \bar{\sigma}, \beta_{0,i}') - F_i(t, \sigma\, \alpha_{0,i}')\right] \\ \qquad + F_i(t, \sigma, \alpha_{0,i}') \text{ if } \alpha_{0,i} \leqslant u_i \leqslant \beta_{0,i} \\[2ex] F_i(t, \sigma, \alpha_{0,i}') + \dfrac{\alpha_{0,i} - u_i}{1 + u_i^2} \text{ if } u_i < \alpha_{0,i}. \end{cases}$$

Since $\alpha_{0,i} < \beta_{0,i}$ for all $t \in I$, $G_i(t, u)$ is well defined. We now consider the modified PBVP

$$-u_i'' = G_i(t, u), u(0) = u(2\pi), u'(0) = u'(2\pi). \quad \text{...(3.3)}$$

Lemma 3.3—Assume that (A_0), (A_1) and (A_3) hold. Then the problem (3.3) has a unique solution u satisfying

$$\alpha_{0,i}(t) \leqslant u_i(t) \leqslant \beta_{0,i}(t),\ t \in I.$$

PROOF : It is clear from the definition that $G_i(t, u)$ is completely continuous and bounded on $I \times E$. Hence we can find a positive number J that depends on α_0 and β_0 such that $\|G(t, u)\| \leqslant J$. Let $X = L_2[0, 2\pi]$ define $Lu_i = -u_i''$ and then $D(L) = \{\varphi \in X: \varphi, \varphi'$ are real valued absolutely continuous on $[0, 2\pi]$, $\varphi'' \in X$, $\varphi(0) = \varphi(2\pi)$ and $\varphi'(0) = \varphi'(2\pi)\}$. Let $\mathscr{N}$ be the nonlinear operator defined by

$$\mathscr{N} u = G(t, u).$$

Then the BVP (3.3) may be translated into the operator equation

$$L u = \mathscr{N} u$$

Notice that X_0, the Kernel of L consists of all constant functions and hence X_i where $X = X_0 \oplus X_i$ is the class of all vector functions whose average on $[0, 2\pi)$ is zero. Also we can define the operators P and H satisfying the conditions of Theorem 2.1 of Kannan and Lakshmikantham[7]. Since G is bounded we can find constants A and B that depend only on α_0, β_0 such that any solution of (2.2) in Theorem 2.1 of Kannan and Lakshmikantham[7] satisfies $\|u_1(t)\| \leqslant A$ and $\|u_i'(t)\| \leqslant B$ for all $t \in I$.

Hence by Theorem 2.1 of Kannan and Lakshmikantham[7] it is enough to find an $R_{0,i} > 0$ such that

$$< \mathscr{N}(u_{0,i} + u_{1,i}),\ u_{0,i} > \ \geqslant 0 \text{ or } \leqslant 0 \qquad \ldots(2.4)$$

for all $|u_{0,i}| = R_{0,i}$ and $|u_{1,i}| \leqslant A$, $|u_{1,i}'| \leqslant B$ on I, $i \in Z^+$. Since X_0 consists of all constants functions and $u_0 \in X_0$, (3.4) can be written as

$$\int_0^{2\pi} G_i(s, R_0 + u_1(s))\, d s \leqslant 0$$

and

$$\int_0^{2\pi} G_i(s, -R_0 + u_1(s))\, ds \geqslant 0$$

choose $R_{0,i} > 0$ large enough so that $R_{0,i} + u_{1,i} > \max_I \beta_{0,i}(t)$ and $-R_{0,i} + u_{1,i} < \min_I \alpha_{1,i}(t)$, Using the definition of G_i and $\left(A_1^* \right)$ we see that

$$\int_0^{2\pi} G_i(s, R_0 + u_1(s))\, d s < \int_0^{2\pi} F_i(s, \bar{\sigma}, \beta_{0,i}'(s))\, ds < 0$$

and

$$\int_0^{2\pi} G_i(s, -R_0 + u_1(s))\, d s > \int_0^{2\pi} F_i(s, \sigma, \alpha_{0,i}'(s))\, d s > 0$$

for any arbitrary but fixed σ, $\bar{\sigma}$ satisfying

$$\alpha_0(t) \leqslant \sigma,\ \bar{\sigma} \leqslant \beta_0(t),\ \sigma_i\ \alpha_0,(t) \text{ and } \bar{\sigma}_i = \beta_{0,i}(t)$$

Hence by Theorem 2,1 of Kannan and Lakshmikantham[7], there exists a solution u to the problem (3.3). Proceeding on the similar lines of that of Lemma 3.1, it is easy to see that $u(t)$ satisfies the inequality

$\alpha_{0,i}(t) \leqslant u_i(t) \leqslant \beta_{0,i}(t),\ t \in I.$ From this it follows that for each $i \in Z^+$, u_i is a solution of

$$-u_i'' = \frac{u_i - \alpha_{0,i}}{\beta_{0,i} - \alpha_{0,i}}\left[F_i\left(t, \bar{\sigma}, \beta'_{0,i}\right) - F_i\left(t, \sigma, \alpha'_{0,i}\right)\right] + F_i\left(t, \sigma, \alpha'_{0,i}\right) \qquad \ldots(3.5)$$

$$u(0) = u(2\pi),\ u'(0) = u'(2\pi)$$

where $\alpha_0(t) \leqslant \sigma, \bar{\sigma} \leqslant \beta_0(t)$, $\sigma_i = \alpha_{0,i}(t)$ and $\bar{\sigma}_i = \beta_{0,i}(t)$. Following the proof of Lemma 2.3 of Rao and Vatsala[15] it can be shown that the solution $u(t)$ of (3.5) is unique.

We now prove our main result of this section.

Theorem 3.1—Let the assumptions $(A_0) - (A_3)$ be satisfied. Then the PBVP (2.1) has a solution u such that $\alpha(t) \leqslant u(t) \leqslant \beta(t)$ and $|u(t)| \leqslant N$ on I, where N depends only on α, β and the Nagume function.

PROOF : Consider the boundary value problem

$$-u_i'' = H_i\left(t, u, u_i'\right),\ u(0) = u(2\pi),\ u'(0) = u'(2\pi) \qquad \ldots(3.6)$$

where

$$H_i\left(t, u, u_i'\right) = \lambda\ F_i\left(t, u, u_i'\right) + (1-\lambda)\, G_i(t, u),\ \lambda \in [0, 1].$$

In view of Lemma 3.2, one can verify that α_0 and β_0 satisfy $\left(A_0^*\right)$, $\left(A_1^*\right)$ and $\left(A_2^*\right)$ with respect to H_i. Using the arguments similar to that of Lemma 3.1, it is easy to show that, if $u_{\lambda,i}$ for $\lambda \in (0, 1)$ is a solution of (3.6) then $\alpha_{0,i}(t) \leqslant u_{\lambda,i}(t)$, $\leqslant \beta_{0,i}(t)$, $t \in I$, $i \in Z^+$. Since H_i satisfies a Nagumo condition, we have $|u'_{\lambda,i}(t)| = C$, $t \in I$ where the constant C is independent of λ. In view of Lemma 3.3, it follows that for $\lambda \in [0,1)$ all possible solutions of (3.6) satisfy $\alpha_0(t) \leqslant u_\lambda(t) \leqslant \beta_0(t)$ and $|u'_{\lambda,i}(t)| \leqslant C$ for all $t \in I$. Also for $\lambda = 0$, the problem (3.6) has a unique

solution. Thus one can choose a bounded, closed convex set B in the (u, u') space $H^1[0, 2\pi) = L_2\ [0, 2\pi]$ such that (3.6) has no solution on the boundary of B for $\lambda \in (0, 1)$ and it has a unique solution in the interior of B for $\lambda = 0$. Hence by Leray-Schauder theory, the problem (3.6) has a solution u for $\lambda = 1$. By Lemma 3.1 we have $\alpha(t) \leqslant u(t) \leqslant \beta(t)$, $t \in I$ and this u is a solution of (2.1) satisfying $\alpha(t) \leqslant u(t) \leqslant \beta(t)$, $t \in I$. Thus (A_2) implies that $|u'(t)| \leqslant N$, $t \in I$ where N is the Nagume constant vector. This completes the proof.

As a special case of Theorem 3.1 we have the following result.

Corollary 3.1—Let the assumptions (A_0), (A_2), (A_3) and (A_5) hold. Further assume that there exist coupled lower and upper quasi-solutions α and β. Then the conclusion of Theorem 3.1 is true.

Remark 3.1 : Note that if α, β are coupled lower and upper quasi-solutions, then (A_1) holds if f has *mq mp* and also that (A_1) implies α, β are coupled lower and upper quasi-solutions of (2.1).

Our results can also be extended to the following infinite system of second order equations with homogeneous Neumann boundary conditions (NBVP for short). That is

$$-u_i'' = f_i\left(t, u, u_i'\right),\ u'(0) = u'(2\pi) = 0 \qquad \ldots(3.7)$$

where

$$f \in C[I \times E \times R, E] \text{ and } i \in Z^+.$$

Theorem 3.2—Let (A_0), (A_2) and (A_3) hold. Further let the following condition hold :

(B) (i) $\alpha'(0) \geqslant 0$, $\alpha'(2\pi) \leqslant 0$ and

$$-\alpha_i''(0) \leqslant f_i\left(t, \sigma, \alpha_i'\right), t \in (0, 2\pi] \text{ for all } \sigma \text{ such that}$$

$$\alpha(t) \leqslant \sigma \leqslant \beta(t) \text{ and } \sigma_i = \alpha_i(t),\ t \in I,\ i \in Z^+$$

(ii) $\beta'(0) \leqslant 0$, $\beta'(2\pi) \geqslant 0$ and

$$-\beta_i'' \geqslant f_i\left(t, \sigma, \beta_i'\right),\ t \in (0, 2\pi] \text{ for all } \sigma$$

such that $\alpha(t) \leqslant \sigma \leqslant \beta(t)$ and $\sigma_i = \beta_i(t)$, $t \in I$, $i \in Z^+$.

Then the problem (3.7) has a solution u such that

$$\alpha(t) \leqslant u(t) \leqslant \beta(t) \text{ and } |u'(t)| \leqslant N \text{ on } I.$$

Where N depends only on α, β and the Nagume function.

The proof is similar to the proof of Theorem (3.1) with appropriate modifications.

4. Monotone Iterative Method

For any $\eta, \mu \in C[I, E]$, $\alpha(t) \leqslant \eta(t), \mu(t) \leqslant \beta(t)$, $t \in I$, we consider the quasilinear PBVP

$$- u_i'' = G_i\ (t, u_i, [u]_p, [u]_q, u_i'),\ u(0) = u(2\pi),\ u'(0) = u'(2\pi) \quad \ldots(4.1)$$

where

$$G_i\ (t, u_i, [u]_p, [u]_q, u_i') = f_i\ (t, \eta_i, [\eta]_p, [u]_q, g(u_i')) - M_i\ (u_i - \eta_i))$$

and

$$g\left(u_i'\right) = \max\left[-\bar{d}_i, \min\left(u_i', \bar{d}_i\right)\right].$$

Notice that G is defined on $I \times [\alpha, \beta] \times R$ and (A_4) is equivalent to

$$\left(A_{4'}^{*}\right) f_i\left(t, u_i, [u]_p, [u]_q, g\left(u_i'\right)\right) - f_i\left(t, v_i [v]_p, [u]_q,\right.$$

$$\left. g\left(u_i'\right)\right) \geqslant - M_i\ (u_i - v_i)$$

for $M_i > 0$, $t \in I$, $\alpha(t) \leqslant v \leqslant u \leqslant \beta(t)$ and $u_i' \in R$.

Relative to the PBVP (4.1) we prove the following lemmas.

Lemma 4.1—Let the assumptions $(A_0) - (A_2)$, (A_4) and (A_5) be satisfied. Then the assumptions (A_1) and (A_2) are true with respect to the PVBP (4.1). That is α, β are also coupled lower and upper quasi-solutions of the PBVP (4.1) and G_i satisfies the modified Nagumo condition relative to α, β.

Proof : Using the arguments of Lemma 3.1 of Lakshmikantham *et al.*[11] it is easy to show that α, β are also coupled lower and upper quasi solutions of the PBVP (4.1). However when (A_2) holds we have

$$| f_i\ (t, u_i, [u]_p, [u]_q, u_i')| \leqslant \begin{cases} h_i\left(|u_i'|\right) \text{ if } |u_i'| \leqslant d_i \\ h_i\ (d_i) \text{ if } |u_i'| > d_i \end{cases} \quad \ldots(4.2)$$

for $t \in I$ and some $d_i > e_i = \dfrac{1}{2\pi} \max\{|\alpha_i(0) - \beta_i(2\pi)|, |\alpha_i(2\pi) - \beta_i(0)|\}$

$\alpha(t) \leqslant u(t) \leqslant \beta(t)$, $u_i' \in R$ and $h_i \in C[[0, \infty), (0, \infty)]$, also

$$\int_{e_i}^{l_i} \frac{s\, ds}{h_i(s)} > \max_I \beta_i(t) - \min_I \alpha_i(t)$$

where

$$l_i = \min \{d_i, N_i\}.$$

From this it follows that any solution $u \in C^2 [I, E]$ of (2.1) satisfies $| u_i' \quad | \leqslant N_i$, $t \in I$.

For $t \in I$, $\alpha (t) \leqslant u (t) \leqslant \beta (t)$ $\alpha, (t) \leqslant \eta$, $\mu \leqslant \beta (t)$ and $u_i' \in R$ we have

$$| M_i (u_i - \gamma_i) | \leqslant M_i \gamma_i$$

where

$$\gamma_i = \max_I \beta_i (t) - \min_I \alpha_i (t)$$

Further

$$| G_i (t, u_i [u]_p, [u]_q, u_i') | = H_i (|u_i' |).$$

where

$$H_i (s) = \begin{cases} h_i (s) + M_i \gamma_i \text{ if } s \leqslant \overline{d_i} \\ h_i (\overline{d_i}) + M_i \gamma_i \text{ if } s > \overline{d_i} \end{cases}$$

Evidently since $H_i (s)$ is a positive constant for $s \geqslant \overline{d_i}$, there exists an $N_i^* > \bar{d}_i$ such that

$$\int_{e_i}^{N_i^*} \frac{s \, ds}{H_i (s)} \geqslant \int_{d_i}^{N_i^*} \frac{s \, ds}{H_i (s)} > \gamma_i$$

and this proves that G_i also satisfies (A_2). Hence the proof is complete.

We now prove a result on existence and uniqueness of solutions of the PBVP (4.1).

Lemma 4.2—Let the assumptions $(A_0) - (A_5)$ hold. Then there exists a solution u of (4.1) such that $\alpha (t) \leqslant u (t) \leqslant \beta (t)$ and $| u_i' (t) | \leqslant \hat{N}_i$ on I. Further more the solution $u (t)$ is unique.

PROOF : By Lemma (4.1) we have all the assumptions $(A_0) - (A_2)$ satisfied with respect to the PBVP (4.1). Hence by Theorem (3.1), there exists a solution $u (t)$ of (4.1) with $\alpha (t) \leqslant u (t) \leqslant \beta (t)$ and $| u_i' (t) | \leqslant \hat{N}_i$ on I. Using the arguments similar to that of Lemma 3.2 of Lakshmikantham *et al.*[11] we can show that the solution $u (t)$ is unique.

Since for every $\eta, \mu \in [\alpha, \beta]$, the PBVP (4.1) has a unique solution u, we define the mapping A by

$$A(\eta, \mu) = u \qquad \ldots(4.3)$$

and study the properties of this mapping in the next lemma.

Lemma 4.3—Under the assumptions of Lemma 4.2, the mapping A defined by (4.3) satisfies the following properties.

(i) $\alpha \leqslant A(\alpha, \beta)$ and $\beta \geqslant A(\beta, \alpha)$

(ii) For $\alpha \subseteq \eta \leqslant \mu \leqslant \beta$. $A(\eta, \mu) \leqslant A(\mu, \eta)$.

PROOF : We shall only prove that $\beta \geqslant A(\beta, \alpha)$ since similar arguments prove that $\alpha \leqslant A(\alpha, \beta)$.

Let $A(\beta, \alpha) = u$, where u is the unique solution of the PBVP (4.1) with $\eta = \beta$ and $\mu = \alpha$. Let $\varphi(t) = u(t) - \beta(t)$. Suppose that the inequality $\varphi(t) \leqslant 0$, $t \in I$ is false. Then there exists a $t_0 \in I$ and an $\epsilon > 0$ such that for some index $k \in Z^+$, we have

$$\varphi_k(t_0) = \epsilon \text{ and } \varphi_i(t) \leqslant \epsilon \text{ for all } t \in I \text{ and } i \in Z^+. \qquad \ldots(4.4)$$

If $t_0 \in (0, 2\pi)$, we have $\varphi_k'(t_0) = 0$ and $\varphi_k''(t_0) \leqslant 0$.

Also

$$g(u_k'(t_0)) = g(\beta_k'(t_0)) = \beta_k'(t_0).$$

At $t = t_0$ using (A_1) and (4.1), we have

$$0 \geqslant \varphi_k''(t_0) = u_k''(t_0) - \beta_k''(t_0)$$

$$\geqslant -G_k(t_0, u_k(t_0), [u(t_0)]_p, [u(t_0)]_q, u_k'(t_0))$$

$$+ f_k(t_0, \beta_k(t_0), [\beta(t_0)]_p, [\alpha(t_0)]_q, \beta_k'(t_0))$$

$$\geqslant M_k \epsilon > 0, \text{ a contradiction.}$$

If $t_0 = 0$, then using the boundary conditions, (A_1) (ii) and (4.4) we obtain

$$\varphi_k(0) = u_k(0) - \beta_k(0) = u_k(2\pi) - \beta_k(2\pi) = \varphi_k(2\pi) = \epsilon$$

$$\varphi_k'(0) \leqslant 0 \text{ and } \varphi_k'(2\pi) \geqslant 0.$$

Also

$$\varphi_k'(0) = u_k'(0) - \beta_k'(0) \geqslant u_k'(2\pi) - \beta_k'(2\pi) = \varphi_k'(2\pi).$$

Hence $\varphi_k'(2\pi) = 0$ and using (A_1) and (4.1) we get

$$\varphi_k''(2\pi) = u_k''(2\pi) - \beta_k^*(2\pi) \geq M_k\,\epsilon > 0 \text{ which is again a contradiction.}$$

To prove (ii) let $\eta, \mu \in [\alpha, \beta]$ such that $\eta \leqslant \mu$. Let $A(\eta, \mu) = x$, $A(\mu, \eta) = y$ and $\psi(t) = x(t) - y(t)$. If the inequality $\psi(t) \leqslant 0$ for $t \in I$ is false, then there exist $t_0 \in I$ and an $\epsilon > 0$ such that for some $k \in Z^+$, we have

$$\psi_k(t_0) = \epsilon \text{ and } \psi_i(t) \leqslant \epsilon \text{ for } t \in I \text{ and } i \in Z^+. \qquad \ldots(4.5)$$

If $t_0 \in (0, 2\pi)$, we have

$$\psi_k'(t_0) = 0 \text{ and } \psi_k''(t_0) \leqslant 0.$$

From (4.5), $\left(A_4^*\right)$ and (A_5) and inview of the definition of G,

$$0 \geq \psi_k''(t_0) = x_k''(t_0) - y_k''(t_0)$$

$$= -f_k(t_0, \eta_k, [\eta]_p, [\mu]_q, g(x_k'(t_0)) + M_k(x_k(t_0) - \eta_k(t_0))$$

$$+ f_k(t_0, \mu_k, [\mu]_p, [\eta]_q, g(y_k'(t_0)) - M_k(y_k(t_0 - \mu_k(t_0))$$

$$\geq M_k(x_k(t_0) - y_k(t_0) > 0, \text{ a contradiction.}$$

If $t_0 = 0$, then from (4.5) and the boundary conditions we obtain $\psi_k(2\pi) = \epsilon$ and $\psi_k'(2\pi) = 0$ and as before we get a contradiction at $t = 2\pi$. This completes the proof The following is the main theorem of this section.

Theorem 4.1—Let the assumptions $(A_0) - (A_5)$ be satisfied. Then there exist monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\alpha_n(t)$ and $\beta_n(t)$ converge uniformly and monotonically to $\rho(t)$ and $r(t)$ respectively on I. Also (ρ, r) are coupled minimal and maximal quasi-solutions of the PBVP (2.1). More precisely, if (x, y) are any coupled quasi-solutions of (2.1) satisfying $\alpha \leqslant x, y \leqslant \beta$, then

$$\alpha = \alpha_0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_n \leqslant \ldots \leqslant \rho \leqslant x, y \leqslant r \leqslant \ldots \leqslant \beta_n$$
$$\leqslant \ldots \leqslant \beta_1 \leqslant \beta_0 = \beta \text{ on } I. \qquad \ldots(4.6)$$

Further more any other solution u of (2.1) satisfying $\alpha(t) \leqslant u \leqslant \beta(t)$ on I also satisfies (4.6) on I.

PROOF : We know from Lemma 4.2 that for any $\eta, \mu \in [\alpha, \beta]$, the PBVP (4.1) has a unique solution $u(t)$ such that $\alpha(t) \leqslant u \leqslant \beta(t)$ and $|u_i'(t)| \leqslant \hat{N}_i$ on I, where $\hat{N}_i$ is the Nagumo constant relative to G. In view of Lemma 4.3 we may define

the sequences $\alpha_n = A(\alpha_{n-1}, \beta_{n-1})$ and $\beta_n = A(\beta_{n-1}, \alpha_{n-1})$, $n = 1, 2, 3, \ldots$ such that $\alpha_0 = \alpha$ and $\beta_0 = \beta$ and $\alpha_n \leqslant \beta_n$ for each n. Since $\alpha_0 \leqslant \alpha_1 \leqslant \beta_1 \leqslant \beta_0$, by induction and the arguments similar to those used in Lemma 4.3, we can establish that $\{\alpha_n\}$, $\{\beta_n\}$ are monotone sequences such that

$$\alpha_0 \leqslant \alpha_1 \leqslant \ldots \leqslant \alpha_n \leqslant \beta_n \leqslant \ldots \beta_2 \leqslant \beta_1 \leqslant \beta_0 \text{ on } I.$$

Where $\alpha_n(t)$ and $\beta_n(t)$ satisfy

$$-\alpha''_{n,i} = f_i(t, \alpha_{n-1,i}, [\alpha_{n-1}]_p, [\beta_{n-1}]_q, g(\alpha'_{n,i}))$$

$$- M_i(\alpha_{n,i} - \alpha_{n-1,i}), \alpha_n(0) = \alpha_n(2\pi), \alpha'_n(0) = \alpha'_n(2\pi) \qquad \ldots(4.7)$$

$$-\beta''_{n,i} = f_i(t, \beta_{n-1,i}, [\beta_{n-1}]_p, [\alpha_{n-1}]_q, g\left(\beta'_{n,i}\right))$$

$$-M_i(\beta_{n,i} - \beta_{n-1,i}), \beta_n(0) = \beta_n(2\pi), \beta'_n(0) = \beta'_n(2\pi) \qquad \ldots(4.8)$$

and

$$|\alpha'_n(t)|, |\beta'_n(t)| \leqslant \hat{N}.$$

From (4.7),

$$-\alpha_{n,i}(t) = \int_0^t \int_0^0 f_i(s, \alpha_{n-1,i}(s), [\alpha_{n-1}]_p, [\beta_{n-1}]_q, g(\alpha'_{n,i}(s)))\, ds\, d\sigma$$

$$\int_0^t \int_0^\sigma M_i(\alpha_{n,i}(s) - \alpha_{n-1,i}(s))\, ds\, d\sigma + C_{n,i}\, t + \lambda_{n,i}.$$

Since for each i, f_i is completely continuous, the sequence $\{\alpha_n\}$ is uniformly bounded and equi-continuous. Thus $\{\alpha_n\}$ contains a subsequence which is uniformly convergent by the Arzela-Ascoli theorem. In view of the fact that $\{\alpha_n\}$ is monotone the full sequence converges uniformly on I. Further more the uniform boundedness of the sequence $\{\alpha''_n\}$ implies that the sequence $\{\alpha'_n\}$ is equi-continuous and uniformly bounded. Thus $\{\alpha'_n\}$ contains a subsequence which is uniformly convergent on I. Similar reasoning for $\{\beta_n\}$. Thus we can find sub-sequences which we again denote by $\{\alpha_n\}$, $\{\beta_n\}$ converging uniformly and monotonically to ρ, r. Where $\rho(t) = \lim_{n\to\infty} \alpha_n(t)$ and $r(t) = \lim_{n\to\infty} \beta_n(t)$, $t \in I$. And this implies that

$$-\rho''_i\, f_i(t, \rho_i, [\rho]_p, [r]_q, g(\rho'_j)), \rho(0) = \rho(2\pi), \rho'(0) = \rho'(2\pi)$$

$$-r''_i = f_i(t, r_i, [r]_p, [\rho]_q, g(r'_i)), r(0) = r(2\pi), r(0) = r'(2\pi) \qquad \ldots(4.9)$$

Following a continuation argument similar to that of [Bernfeld and Lakshmtkantham[1] p. 32], one can prove that ρ and r are actually coupled quasi-solutions of th PBVP (2.1). If (u, v) are any coupled solutions of (2.1) such $u, v \in [\alpha, \beta]$ $t \in I$ and $|u'|, |v'| \leqslant \hat{N} \leqslant \bar{d}$ on I, employing induction principle and the arguments similar to those used earlier, it can be shown that $\alpha_n \leqslant u, v \leqslant \beta_n$ on I. Hence we have $\rho \leqslant u, v \leqslant r$ on I, proving that ρ, r are coupled minimal and maximal quasi-solutions of the PBVP (2.1).

Since any solution u of (2.1) satisfying $\alpha \leqslant u \leqslant \beta$ on I may be regarded as (u, u) coupled quasi-solution of (2.1), we also have $\rho \leqslant u \leqslant r$ on I. This completes the proof of the theorem.

5. Uniqueness Result

We now present a result on the uniqueness of solutions of the PBVP (2.1).

Theorem 5.1—Assume (A_0) — (A_3). In addition for each $i \in Z^+$, there exists a constant $L_i > 0$ such that

$$(u_i - v_i)\,[\,f_i\,(t, u, u_i') - f_i\,(t, v, v_i')] \leqslant -L_i\,(u_i - v_i)^2 \qquad \ldots(5.1)$$

whenever $\alpha\,(t) \leqslant u, v \leqslant \beta\,(t)$ $t \in I$ and $u_i' - v_i' = 0$. Then the PBVP (2.1) has a unique solution $u\,(t)$ satisfying $\alpha\,(t) \leqslant u\,(t) \leqslant \beta\,(t)$ on I.

Proof : By Theorem 3.1 we know that the PBVP (2.1) has a solution. If possible, let u and v be two solutions for the PBVP (2.1) satisfying $\alpha\,(t) \leqslant u\,(t)$, $v\,(t) \leqslant \beta\,(t)$ for $t \in I$.

We define

$$\mu_i\,(t) = (u_i\,(t) - v_i\,(t))^2$$

and observe that $\mu_i\,(0) = \mu_i\,(2\pi)$, $\mu_i'\,(0) = \mu_i'\,(2\pi)$ for each $i \in Z^+$

and

$$\mu_i''\,(t) = -2\,(u_i\,(t) - v_i\,(t))[\,f_i\,(t, u, u_i') - f_i\,(t, v, v_i')]$$

$$+\,2\,(u_i'\,(t) - v_i'\,(t))^2 \qquad \ldots(5.2)$$

We claim that $\mu_i\,(t) \equiv 0$ for all $i \in Z^+$ on I If not there exists a $t_0 \in I$ and an $\epsilon > 0$ such that for some $k \in Z^+$

$$\mu_k\,(t_0) = \epsilon \text{ and } \mu_i\,(t) \leqslant \epsilon \text{ for all } t \in I,\ i \in Z^+. \qquad \ldots(5.3)$$

If $t_0 \in (0, 2\pi)$, we have

$$\mu_k'\,(t_0) = 0 \text{ and } u_k'' \leqslant 0.$$

Thus from (5.2) and (5.1) we have

$$0 \geqslant \mu_k(t_0) = -2(u_k(t_0) - v_k(t_0))\,[f_k(t_0, u, u_k'(t_0))$$
$$- f_k(t_0, v, v_k'(t_0))] + 2(u_k'(t_0) - v_k'(t_0))^2$$
$$\geqslant 2L_k \mu_k(t_0) > 0, \text{ a contradiction.}$$

If $t_0 = 0$, we obtain $\mu_k(0) = \epsilon = \mu_k(2\pi)$ and $\mu_k'(0) \leqslant 0$ and $\mu_k'(2\pi) \geqslant 0$. However $\mu_k'(0) = \mu_k'(2\pi)$ and hence $\mu_k'(0) = \mu_k'(2\pi) = 0$ and consequently $\mu_k''(\lambda) \leqslant 0$ for $\lambda = 0, 2\pi$.

Also

$$0 \geqslant \mu_k''(\lambda) = 2L_k\, \mu_k(\lambda) > 0, \text{ for } \lambda = 0, 2\pi$$

which is again a contradiction. Hence the proof is complete.

Corollary 5.1—Assuming the conditions of Theorem 4.1 and the hypothesis (5.1) one may conclude the existence of a unique solution for the PBVP (2.1). In this case, if suffices to show that the coupled minimal and maximal quasi-solutions $\rho(t)$ and $r(t)$ are identical. This can be accomplished by defining the function

$$\mu_i(t) = (\rho_i(t) - r_i(t))^2 \text{ for each } i \in Z^+ \text{ and } t \in I$$

and proceeding along the lines of the proof of Theorem 5.1.

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ON THE SETS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS AND THE REGGE, BARGMANN-SHELEPIN ARRAYS FOR THE 3–J AND 6–J COEFFICIENTS

K. SRINIVASA RAO AND V. RAJESWARI

The Institute of Mathematical Sciences, Madras 600113

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The connection between the $3-j$ and the $6-j$ coefficients to a set of six $\underset{3-2}{F}(1)$ s and a set of three (or, equivalently a set of four) $\underset{4-3}{F}(1)$ s, respectively, is used to obtain sets of Regge 3×3 and Bargmann—Shelepin 4×3 symbols. Closed form expressions are obtained for the polynomial zeros of degree n of these coefficients.

INTRODUCTION

In literature, classical symmetries of the $3-j$ and the $6-j$ coefficients are 12 and 24 in number. By relating the $3-j$ coefficient to a 3×3 magic square symbol, Regge[1] showed that it has 72 symmetries. Bargmann[2] and Shelepin[3] related the $6-j$ coefficient to a 4×3 symbol, which exhibits all the 144 symmetries (including the classical symmetries) discovered by Regge[4]. We have shown that sets of ${}_{p+1}F_p$ (1) s are necessary and sufficient to account for all the known symmetries of the $3-j$ and the $6-j$ coefficients—explicitly, while a set of six ${}_3F_2$ (1) s represent the $3-j$ coefficient[5], either a set I of three or an equivalent set II of four ${}_4F_3$ (1) s represent the $6-j$ coefficient[6,7]. Here we establish a connection between these sets of generalized hypergeometric functions and sets of Regge, Bargmann-Shelepin symbols for the $3-j$ and the $6-j$ coefficients, respectively. Using these, closed form expressions have been obtained for the polynomial zeros of degree n of the $3-j$ and $6-j$ coefficients, which have been the subject of detailed study-especially when $n = 1$ or 2—by several authors in recent years.

CLOSED FORM EXPRESSIONS

The $3-j$ coefficient has been defined by Wigner[8] as :

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m^2 & m_3 \end{pmatrix} = \delta(m_1 + m_2 + m_3, 0)\,(-1)^{j_1 - j_2 - m_3}\,\Delta(j_1\, j_2\, j_3) \prod_{i=1}^{3} \{(j_i + m_i)!\,(j_i - m_i)!\}^{1/2} \times \sum_z (-1)^z \{z! \prod_{k=1}^{2} \{z - \alpha_k)! \prod_{l=1}^{3} (\beta_l - z)!\}^{\ 1} \qquad \ldots (1)$$

where

$$\max(\alpha_1, \alpha_2) \leqslant z \leqslant \min(\beta_1, \beta_2, \beta_3)$$

$$\alpha_1 = j_1 - j_3 + m_2,\ \alpha_2 = j_2 - j_3 - m_1$$

$$\beta_1 = j_1 - m_1,\ \beta_2 = j_2 + m_2,\ \beta_3 = j_1 + j_2 - j_3. \qquad ...(2)$$

and

$$\Delta(x\,y\,z) = [(-x + y + z)!\,(x - y + z)!\,(x + y - z)!/(x + y + z + 1)!]^{1/2} \qquad ...(3)$$

and $\delta(x, y)$ is the Kronecker delta function. Regge[1] discovered new symmetries by associating the $3 - j$ coefficient with a magic 3×3 square symbol :

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{vmatrix} -j_1 + j_2 + j_3 & j_1 - j_2 + j_3 & j_1 + j_2 - j_3 \\ j_1 - m_1 & j_2 - m_2 & j_3 - j_3 \\ j_1 + m_1 & j_2 + m_2 & j_3 + m_3 \end{vmatrix} = \|R_{ik}\| \qquad ...(4)$$

and asserted that the 3×3 square symbol represents the invariance of the $3-j$ coefficient to 3! column and 3! row permutations and to a reflection about its diagonal. Thus, Regge[1] established the existence of a 72-element symmetry group, comprising the well-known classical symmetries (which arise due to column permutations and to the space reflection : $m_i \to -m_i$ arising due to the interchange of the second and third rows of (4)) and six new symmetries known as Regge symmetries of the $3-j$ coefficient.

It has been shown by one of us (Srinivasa Rao[5]) that the $3-j$ coefficient can be represented by a set of six ${}_3F_2(1)$ s :

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta(m_1 + m_2 + m_3, 0)\,(-1)^{\sigma(pqr)} \prod_{i,k=1}^{3} \{R_{ik}!/(J+1)!\}^{1/2}$$

$$\times [\Gamma(1 - A, 1 - B, 1 - C, D, E)]^{-1}\ {}_3F_2(A, B, C; D, E; 1) \qquad ...(5)$$

where

$$A = -R_{2p},\ B = -R_{3q},\ C = -R_{1r},\ D = R_{3r} - R_{2p} + 1,$$

$$E = R_{2r} - R_{3q} + 1,\ \Gamma(x, y, \ldots) = \Gamma(x)\,\Gamma(y)\ldots,$$

$$J = j_1 + j_2 + j_3 \qquad ...(6)$$

and

$$\sigma(pqr) = \begin{cases} R_{3p} - R_{2q} & \text{for even permutations} \\ R_{3p} - R_{2q} + J & \text{for odd permutations} \end{cases}$$

for all permutations of $(p\ q\ r) = (123)$. Since each one of the six ${}_3F_2\,(1)\,s$ represents only 12 symmetries arising from its invariance to 3! numerator and 2! denominator parameter permutations, the set of six ${}_3F_2\,(1)\,s$ is necessary and sufficient to account for all the 72 symmetries of the $3-j$ coefficient.

Using the properties of the elements of the 3×3 square symbol :

$$R_{lp} + R_{mp} = R_{nq} + R_{nr} \qquad \ldots(7)$$

for (lmn) and (pqr) being (123) cyclically, and the defining relations (6) for the numerator and denominator parameters, one can easily show that :

$$\|R_{ik}\| = \begin{vmatrix} -B+D+1 & -A+E-1 & -C \\ -A & -C+D-1 & -B+E-1 \\ -C+E-1 & -B & -A+D-1 \end{vmatrix}. \qquad \ldots(8)$$

From (8) it is straightforward to obtain the closed form expression :

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} (-A-C+E-1)/2 & (-B-C+D-1)/2 \\ (A-C+E-1)/2 & (-B+C-D+1)/2 \end{pmatrix}$$
$$\times \begin{matrix} (-A-B+D+E-2)/2 \\ (-A+B+D-E)/2 \end{matrix} \Bigg) \qquad \ldots(9)$$

The parameters A and B are negative integers by definition. So, if we let $P = -A$ and $Q = -B$ and set $C = -1$, we get, for the polynomial zeros of degree one the expression :

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} (P+E)/2 & (Q+D)/2 \\ (-P+E)/2 & (Q-D)/2 \end{pmatrix}$$
$$\times \begin{matrix} (P+Q+D+E-2)/2 \\ (P-Q+D-E)/2 \end{matrix} \Bigg) \qquad \ldots(10)$$

with the constraint equation :

$$PQ = DE \qquad \ldots(11)$$

which is a homogeneous multiplicative Diophantine equation (see, Bell[9]) of degree 2 whose complete solutions can be expressed in terms of four parameters. Identifying the four parameter solution to be :

$$P = ab,\ Q = cd,\ D = bd \text{ and } E = ac \qquad \ldots(12)$$

and substituting it in (10), we obtain the result given by Brudno[10]. The polynomial zeros of degree n arise due to the truncation of the ${}_3F_2\,(1)$ series (5). By setting anyone of the numerator parameters, (say C) to $-n$ and equating the sum of the $(n+1)$ terms to zero, one obtains the constraint equation which must be satisfied by the

numerator and denominator parameters of the ${}_3F_2$ (1) for realizing the polynomial zeros of degree n.

The $6-j$ coefficient has been expressed by Regge[4] to be :

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = N \sum_p (-1)^p (p+1)! \{ \prod_{i=1}^{4} (p-\alpha_i)! \prod_{j=1}^{3} (\beta_j - p)! \}^{-1} \quad ...(13)$$

with

$$N = \Delta(a\,b\,e)\,\Delta(c\,d\,e)\,\Delta(a\,c\,f)\,\Delta(b\,d\,f)$$

$$\alpha_1 = a+b+e,\ \alpha_2 = c+d+e,\ \alpha_3 = a+c+f,\ \alpha_4 = b+d+f$$

$$\beta_1 = a+b+c+d,\ \beta_2 = a+d+e+f,\ \beta_3 = b+c+e+f$$

and $\max(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leqslant p \leqslant \min(\beta_1, \beta_2, \beta_3)$. The 144 symmetries of the $6-j$ coefficient arise due to the invariance of (13) to the 4! permutations of the α's and the 3! permutations of the β's. This is explicit in the notation of Bargmann[2] and Shelepin[3] wherein :

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = \begin{Vmatrix} \beta_1-\alpha_1 & \beta_2-\alpha_1 & \beta_3-\alpha_1 \\ \beta_1-\alpha_2 & \beta_2-\alpha_2 & \beta_3-\alpha_2 \\ \beta_1-\alpha_3 & \beta_2-\alpha_3 & \beta_3-\alpha_3 \\ \beta_1-\alpha_4 & \beta_2-\alpha_4 & \alpha_3-\alpha_4 \end{Vmatrix} = \|R_{ik}\| \quad ...(14)$$

which is invariant to 4! row permutations and 3! column permutations. The elements of $\|R_{ik}\|$ satisfy the 18 relations :

$$R_{kk} + R_{mn} = R_{kn} + R_{mk}$$

and

$$R_{4k} + R_{mn} = R_{4n} + R_{mk} \quad ...(15)$$

for $k \neq m$ and $k \neq n$ and k, m or n being 1, 2 or 3. Equivalently, every 2×2 co-factor in (14), say $\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$ satisfies the condition : $\alpha + \delta = \beta + \gamma$.

We[6] have shown that the $6-j$ coefficient can be represented by a set I of three ${}_4F_3$ (1) s as :

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = (-1)^{E+1} N\,\Gamma(1-E)\,\{\Gamma(1-A, 1-B, 1-C, 1-D, F, G)\}^{-1} \times {}_4F_3(A, B, C, D;\ E, F, G; 1) \quad ...(16)$$

where

$$A = -R_{1p},\ B = -R_{2p},\ C = -R_{3p},\quad D = -R_{4p},$$

$$E = -R_{1p} - R_{2p} - R_{3q} - R_{4r} - 1,\quad F = R_{3q} - R_{3p} + 1,\ G = R_{4r} - R_{4p} + 1 \quad ...(17)$$

for $(p\ q\ r) = (123)$ cyclic; and use has been made of (15) in arriving at (17). It is now possible to express the standard Bargmann-Shelepin 4×3 symbol in terms of the numerator and denominator parameters of the set I of ${}_4F_3(1)$ s, using (15) again, as:

$$\left\{ \begin{matrix} a & b & e \\ d & c & f \end{matrix} \right\} = \left\| \begin{matrix} -A & F-A-1 & G-A-1 \\ -B & F-B-1 & G-B-1 \\ -C & F-C-1 & G-D-1 \\ -D & F-D-1 & G-D-1 \end{matrix} \right\| = \|R_{ik}\| \qquad \text{...(18)}$$

where the negative denominator parameter E does not appear in (18) and the 4×3 symbol in (18) exhibits only 48 symmetries which arise due to the invariance of the $6-j$ coefficient to 4! row permutations and 2! column permutations in $\|R_{ik}\|$. The set I of three different 4×3 symbols which exist due to the substitution $(p\ q\ r) = (123)$ cyclically in (17) account for the 144 symmetries exhibited by (14). It is now straightforward to obtain the closed form expression for the $6-j$ coefficient :

$$\left\{ \begin{matrix} a & b & e \\ d & c & f \end{matrix} \right\} = \left\{ \begin{matrix} (G-B-D-1)/2 & (F-B-C-1)/2 & (F+G-C-D-2)/2 \\ (G-A-C-1)/2 & (F-A-D-1)/2 & (F+G-A-B-2)/2 \end{matrix} \right\} \qquad \text{...(19)}$$

Setting one of the numerator parameters of the ${}_4F_3(1)$ to -1 (say, $D = -1$) and replacing the negative parameters A, B and C by $-v$, $-w$, and $-u$, respectively and letting $F = y$ and $G = x$, we get for the r. h. s. of (19):

$$\left\{ \begin{matrix} (x+w)/2 & (y+u+w-1)/2 & (x+y+u-1)/2 \\ (x+u+v-1)/2 & (y+v)/2 & (x+y+v+w-2)/2 \end{matrix} \right\} \qquad \text{...(20)}$$

which is a symmetry of the parametric solution of Brudno and Louck[11], in the notation of Srinivasa Rao *et al.*[12] for the polynomial zeros of degree 1.

It was shown[7] that the $6-j$ coefficient can also be represented by a set II of four ${}_4F_3(1)$ s as :

$$\left\{ \begin{matrix} a & b & e \\ d & c & f \end{matrix} \right\} = (-1)^{A'-2} N\, \Gamma(A')\, [\Gamma(1-B', 1-C', 1-D', E', F', G')]^{-1}$$

$$\times\ {}_4F_3(A', B', C', D'; E', F', G'; 1). \qquad \text{...(21)}$$

where, using (15), the numerator and denominator parameters can be shown to be:

$$A' = R_{q2} + R_{r1} + R_{s3} + 2,\ B' = -R_{p1},\ C' = -R_{p2},\ D' = -R_{p3}$$

$$E' = R_{q1} - R_{p1} + 1,\ F' = R_{r1} - R_{p1} + 1,\ G' = R_{s1} - R_{p1} + 1 \qquad \text{...(22)}$$

for $(p\,q\,r\,s) = (1234)$ cyclically. The 4×3 symbol for this set II of ${}_4F_3\,(1)$ s can be written as :

$$\left\{ \begin{matrix} a & b & e \\ d & c & f \end{matrix} \right\} \left\| \begin{matrix} -B' & -C' & -D' \\ E'-B'-1 & E'-C'-1 & E'-D'-1 \\ F'-B'-1 & F'-C'-1 & F'-D'-1 \\ G'-B'-1 & G'-C'-1 & G'-D'-1 \end{matrix} \right\| = \| R'_{ik} \| \qquad \ldots(23)$$

where the positive numerator parameter A' does not appear in (23) and this 4×3 symbol exhibits only 36 of the 144 symmetries of the $6-j$ coefficient which arise due to its invariance to 3! column (or all permutations of B', C', D') and 3! row permutations (or all permutations of E', F', G') of $\|R'_{ik}\|$. The set II of four $\|R'_{ik}\|$s which arise due to the substitution $(p\;q\;r\;s) = (1234)$ cyclically in (22) accounts for the 144 symmetries of the $6-j$ coefficient. As in the case of the set I of ${}_4F_3\,(1)$ s, in the case of this set II of ${}_4F_3\,(1)$ s also we obtain a closed from expression :

$$\left\{ \begin{matrix} a\,b\,e \\ d\,c\,f \end{matrix} \right\} = \left\{ \begin{matrix} (E'+G'-B'-C'-2)/2 & (E'+F'-B'-D'-2)/2 & (F'+G'-C'-D'-2)/2 \\ (F'-B'-C'-1)/2 & (G'-B'-D'-1)/2 & (E'-C'-D'-1)/2 \end{matrix} \right\} \qquad \ldots(24)$$

Setting one of the numerator parameters of the ${}_4F_3\,(1)$ to -1 (say, $D' = -1$) and replacing the negative parameters B' and C' by $-x$ and $-y$, respectively, and letting E', F', and G' be v, u and w, we get for the r.h.s of (24):

$$\left\{ \begin{matrix} (x+y+v+w-2)/2 & (x+u+v-1)/2 & (y+u+w-1)/2 \\ (x+y+u-1)/2 & (x+w)/2 & (x+v)/2 \end{matrix} \right\} \qquad \ldots(25)$$

which is a symmetry of (20). We have shown elsewhere[12] that the polynomial zeros of degree one of the $6-j$ coefficient are obtained when the parameters in (25) are subject to the condition :

$$x\,y\,z = u\,v\,w \qquad \ldots(26)$$

or equivalently, $A\,B\,C = E\,F\,G$, $D = -1$ in the case of (19) and $A'\,B'\,C' = E'\,F'\,G'$, $D' = -1$ in the case of (24) which is a multiplicative Diophantine equation of degree 3 subject to the constraint :

$$z = x + y + u + v + w. \qquad \ldots(27)$$

Obviously, polynomial zeros of degree n arise when the sum of the first $n + 1$ terms of the ${}_4F_3\,(1)$ occurring in (19) or (24) adds to zero. We could use either (19) or (24) to generate the complete set of zeros of degree n.

The polynomial zeros of degree 2 of the $3-j$ and the $6-j$ coefficients in terms of these closed form expressions have been studied by Louck[13] *et. al.* using their connection to Pell's equation. However their study does not lead to all the polynomial zeros of degree 2 of the $6-j$ coefficient. Simple algorithms based on the principle of factorization of integers have been proposed by Srinivasa Rao and Chiu[14] to obtain all the polynomial zeros of degree 2 of the $3-j$ and the $6-j$ coefficients.

In conclusion, we have shown in this article the connection between sets of ${}_{p+1}F_p$ (1) *s* and sets of Regge or Bargmann-Shelepin symbols for the $3-j$ and the $6-j$ coefficients. This led us to closed form expressions for the polynomial zeros of degree n of these coefficients. As n increases, the complexity of the constraint equation which has to be satisfied by the parameters in the closed from expressions-or, the numerator and dedominator parameters of the ${}_{p+1}F_p$ (1) *s*—increases. At present detailed studies have been made only of the polynomial zeros of degree 1 and 2 of the $3-j$ and $6-j$ coefficients.

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EIGENVALUE APPROACH TO LINEAR MICROPOLAR ELASTICITY

R. K. MAHALANABIS AND J. MANNA

Department of Mathematics, Jadavpur University, Calcutta 700032

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In this paper the basic equations of linear micropolar elasticity in polar coordinates are arranged in the form of vector matrix differential equation in the Hankel transform domain. The problem is then converted to an algebraic eigenvalue yroblem and solved in the same domain. It is seen that the results obtained by eigenvalue approach are in full agreement with those of other researchers. Further it seems that this approach is more elegant and it is believed that this technique has not been applied earlier by any researcher to solve the fundamental equations of linear micropolar elasticity.

INTRODUCTION

In recent years a detailed exposition of the linear theory of micropolar elasticity has been given by Kuvchinski and Aero[10], Eringen and Suhabi[9], Eringen[8], Nowacki[12] etc.. Nowacki[11] has shown that the equations of motion of linear micropolar theory in polar coordinates can be decomposed into two mutually independent sets of three equations each in case of axisymmetric problems. Dhaliwal and Chowdhury[5] have solved the set (2.6) for the axisymmetric Reissner—Sagoci problem and the solution is obtained by the classical method. Dhaliwal[7] has solved the set (2.7) for the axisymmetric Baussinisq problem. Das *et al*.[2,3] have applied recently the eigenvalue approach in solving the basic equations of thermoelasticity and extended the approach to magneto-thermoelasticity. They have solved the basic equations by representing them to single vector matrix differential equation and converting finally to an algebraic eigenvalue problem.

In this paper we apply the technique of eigenvalue to solve the axisymmetric equations of linear micropolar elasticity. It is believed that none of the previous investigators have applied this approach in solving the problems of linear micropolar theory. Here the basic axisymmetric eqns. (2.6) and (2.7) are presented in terms of single vector-matrix differential equations in sections 3 and 4 respectively. These lead to eigenvalue problems (3.8) and (4.4) respectively for the sets (2.6) and (2.7) and these are solved for displacements in the Hankel transform domain. The characteristic equation of (2.6) gives repeated roots while the set (2.7) gives real distinct roots i. e. real distinct eigenvalues. The general solution for distinct roots is obtained by usual

procedure (see appendix A), while for repeated roots the solution is obtained following the procedure of Das *et al.*[2] (see Appendix A). Further, solution for the half-space is also obtained. It is also observed that the solution obtained by Dhaliwal[6,7] for sets (2.6) and (2.7) are in full agreement with those obtained by eigenvalue approach.

2. The Basic Equations

The equations of motion and other basic equations for a homogeneous isotropic centrosymmetric linear elastic body occupying a region R (vide Dhaliwal[7]), are given by

$$\left.\begin{aligned} \sigma_{ji,j} + \rho X_i &= \rho \ddot{u}_i \\ \mu_{ji,j} + \epsilon_{ijk}\,\sigma_{jk} + J Y_i &= J\,\ddot{\omega}_i \end{aligned}\right\} \quad \ldots(2.1)$$

and the kinematic relations are

$$\left.\begin{aligned} \beta_{ij} &= \omega_{j,i} \\ \gamma_{ij} &= u_{j,i} + \epsilon_{kji}\,\omega_k \end{aligned}\right\} \quad \ldots(2.2)$$

the linear constitutive law being

$$\left.\begin{aligned} \sigma_{ij} &= \lambda\,\gamma_{kk}\,\delta_{ij} + 2\mu\gamma_{(ij)} + 2\alpha\,\gamma_{[ij]} \\ \mu_{ij} &= \beta\beta_{kk}\,\delta_{ij} + 2\gamma\,\beta_{(ij)} + 2\epsilon\,\beta_{[ij]} \end{aligned}\right\} \quad \ldots(2.3)$$

where σ_{ij} are the stress tensor components; μ_{ij} the couple stress tensor components; u_i the displacement field components; ω_i the rotational field components; X_i the body force components; Y_i the body couple components; γ_{ij} the strain tensor components; β_{ij} the curvature twist tensor components; ϵ_{ijk} the unit antisymmetric tensor; [ ] and ( ) indicate respectively the skew symmetric and symmetric part of a tensor; λ, μ, α, β, γ, ϵ, are the elastic constants of the micropolar material; ρ is the density; J the rotational inertia; and the dot (.) denotes the derivatives with respect to time.

Substituting (2.2) and (2.3) in (2.1) a set of six differential equations are obtained and these equations are presented in the vector form as :

$$\left.\begin{aligned} &\text{(i)} \quad (\lambda + 2\mu)\,\nabla\nabla.\,\mathbf{u} - (\mu + \alpha)\,\nabla \times \nabla \times \mathbf{u} + 2\alpha\,\nabla \times \boldsymbol{\omega} + \rho x = \rho\ddot{\mathbf{u}} \\ &\text{(ii)} \quad (\beta + 2\gamma)\,\nabla\nabla.\,\boldsymbol{\omega} - (\gamma + \epsilon)\,\nabla \times \Delta \times \boldsymbol{\omega} + 2\alpha\,\nabla \times \mathbf{u} - 4\alpha\boldsymbol{\omega} + JY = J\ddot{\boldsymbol{\omega}} \end{aligned}\right\} \quad \ldots(2.4)$$

Here we observe that the material constant α is responsible for a coupling of a displacement and micro rotation fields. Though these equations are coupled, they are independent in the case, when $\alpha = 0$. In this case eqn. (2.4); (i) reduces to displacement equations of motion of classical elasticity and eqn. (2.4) (ii) describes a hypothetical elastic body in which only rotation occurs. When $\alpha \to \infty$, the couple-stress theory conditions

$$\boldsymbol{\omega} = \tfrac{1}{2}\,\nabla \times \mathbf{u}$$

is obtained.

Now to solve the static problem with no body forces, we take $X = Y = \ddot{u} = \ddot{\omega} = 0$ and the cylindrical polar coordinates (r, φ, z) is introduced. Equations (2.4) now assumes the forms

$$(\mu + \alpha)\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\varphi}{\partial \varphi}\right) + (\lambda + \mu - \alpha)\frac{\partial e}{\partial r}$$

$$+ 2\alpha\left[\frac{1}{r}\frac{\partial \omega_z}{\partial \varphi} - \frac{\partial \omega_z}{\partial z}\right] = 0$$

$$(\mu + \alpha)\left(\nabla^2 u_\varphi - \frac{u_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \varphi}\right) + (\lambda + \mu - \alpha)\frac{1}{r}\frac{\partial e}{\partial \varphi}$$

$$+ 2\alpha\left[\frac{\partial \omega_r}{\partial z} - \frac{\partial \omega_z}{\partial r}\right] = 0$$

$$(\mu + \alpha)\nabla^2 u_z + (\lambda + \mu - \alpha)\frac{\partial e}{\partial z} + 2\alpha\frac{1}{r}\left[\frac{\partial}{\partial r}(r\,\omega_\varphi) - \frac{\partial \omega_r}{\partial \varphi}\right]$$

$$= 0$$

$$(\gamma + \epsilon)\left(\nabla^2 \omega_r - \frac{\omega_r}{r^2} - \frac{2}{r^2}\frac{\partial \omega_\varphi}{\partial \varphi}\right) - 4\alpha\,\omega_r + (\beta + \gamma - \epsilon)$$

$$\times \frac{\partial \psi}{\partial r} + 2\alpha\left(\frac{1}{r}\frac{\partial u_z}{\partial \varphi} - \frac{\partial u_\varphi}{\partial z}\right) = 0$$

$$(\gamma + \epsilon)\left(\nabla^2 \omega_\varphi - \frac{\omega_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial \omega_r}{\partial \varphi}\right) - 4\alpha\,\omega_\varphi + (\beta + \gamma - \epsilon)\frac{\partial \chi}{\partial \varphi}$$

$$+ 2\alpha\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) = 0$$

$$(\gamma + \epsilon)\nabla^2 \omega_z - 4\alpha\,\omega_z + (\beta + \gamma - \epsilon)\frac{\partial \psi}{\partial z} + 2\alpha\frac{1}{r}\left[\frac{\partial}{\partial r}(r\,u_\varphi)\right.$$

$$\left. - \frac{\partial u_r}{\partial \varphi}\right] = 0 \qquad \ldots(2.5)$$

where

$$e = \frac{1}{r}\frac{\partial}{\partial r}(r\,u_r) + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z}$$

$$\chi = \frac{1}{r}\frac{\partial}{\partial r}(r\,\omega_r) + \frac{1}{r}\frac{\partial \omega_\varphi}{\partial \varphi} + \frac{\partial \omega_z}{\partial z} \qquad \ldots(2.6)$$

The case in which the vectors of displacement u and rotation $\vec{\omega}$ depend only on the coordinates r, z and as such the equations (2.5) are decomposed into two mutually independent set of equations, viz.

$$\left.\begin{aligned}
&\text{(a)}\quad (\mu+\alpha)\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) + (\lambda+\mu-\alpha)\frac{\partial e}{\partial r} - 2\alpha\frac{\partial \omega_\varphi}{\partial z} = 0\\
&\text{(b)}\quad (\mu+\alpha)\nabla^2 u_z + (\lambda+\mu-\alpha)\frac{\partial e}{\partial z} + 2\alpha\frac{1}{r}\frac{\partial}{\partial r}(r\omega_\varphi) = 0\\
&\text{(c)}\quad (\gamma+\epsilon)\left(\nabla^2 \omega_\varphi - \frac{\omega_\varphi}{r^2}\right) - 4\alpha\,\omega_\varphi + 2\alpha\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) = 0
\end{aligned}\right\} \quad \ldots (2.6)$$

$$\left.\begin{aligned}
&(\mu+\alpha)\left(\nabla^2 u_\varphi - \frac{u_\varphi}{r^2}\right) + 2\alpha\left(\frac{\partial \omega_r}{\partial z} - \frac{\partial \omega_z}{\partial r}\right) = 0\\
&(\gamma+\epsilon)\left(\nabla^2 \omega_r - \frac{\omega_r}{r^2}\right) - 4\alpha\,\omega_r + (\beta+\gamma-\epsilon)\frac{\partial \psi}{\partial r}\\
&- 2\alpha\frac{\partial u_\varphi}{\partial z} = 0 \quad (\gamma+\epsilon)\nabla^2\omega_2 - 4\alpha\omega_2 + (\beta+\gamma-\epsilon)\frac{\partial \psi}{\partial r}\\
&+ 2\alpha\frac{1}{r}(r\,u_\varphi) = 0
\end{aligned}\right\} \quad \ldots (2.7)$$

where

$$e = \frac{1}{r}\frac{\partial}{\partial r}(r\,u_r) + \frac{\partial u_z}{\partial z}$$

$$\chi = \frac{1}{r}\frac{\partial}{\partial r}(r\,\omega_r) + \frac{\partial \omega_z}{\partial z}$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

3. Solution of Equations (2.6)

Here only the set (2.6) are considered. The following state of force-stress σ and couple stress μ are being ascribed to the displacement vector $\mathbf{u} = (u_r, 0, u_z)$ and rotation vector $\boldsymbol{\omega} = (0, \omega_\varphi, 0)$:

$$\sigma = \begin{vmatrix} \sigma_{rr} & 0 & \sigma_{rz} \\ 0 & \sigma_{\varphi\varphi} & 0 \\ \sigma_{zr} & 0 & \sigma_{zz} \end{vmatrix}, \quad \mu = \begin{vmatrix} 0 & \mu_{r\varphi} & 0 \\ \mu_{\varphi r} & 0 & \mu_{\varphi z} \\ 3 & \mu_{z\varphi} & 0 \end{vmatrix} \quad \ldots (3.1)$$

where the particular components of stress-tensor have the following forms after using the relation (2.3)

$$\sigma_{rr} = 2\mu\frac{\partial u_r}{\partial r} + \lambda e,$$

$$\sigma_{\varphi\varphi} = 2\mu\frac{u_r}{r} + \lambda e$$

$$\sigma_{zz} = 2\mu\frac{\partial u_z}{\partial z} + \lambda e$$

$$\sigma_{rz} = \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) - \alpha\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) + 2\alpha\,\omega_\varphi$$

$$\sigma_{zr} = \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) + \alpha\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) - 2\alpha\,\omega_\varphi$$

$$\mu_{r\varphi} = \gamma\left(\frac{\partial \omega_\varphi}{\partial r} - \frac{\omega_\varphi}{r}\right) + \epsilon\left(\frac{\partial \omega_\varphi}{\partial r} + \frac{\omega_\varphi}{r}\right)$$

$$\mu_{\varphi r} = \gamma\left(\frac{\partial \omega_\varphi}{\partial r} - \frac{\omega_\varphi}{r}\right) - \epsilon\left(\frac{\partial \omega_\varphi}{\partial r} + \frac{\omega_\varphi}{r}\right)$$

$$\mu_{\varphi z} = (\gamma - \epsilon)\,\frac{\partial\,\omega_\varphi}{\partial z},\quad \mu_{z\varphi} = (\gamma + \epsilon)\,\frac{\partial\,\omega_\varphi}{\partial z} \qquad \text{...(3.2)}$$

In the system of eqns. (2.6), three mutually independent functions u_r, u_z and ω_φ are involved. Multiplying (2.6) by $J_0(\xi r)$ and (2.6a,c) by $J_1(\xi r)$ and integrating between the limits 0 to ∞, we find that the system of partial differential equations (2.6) reduces to the following system of ordinary differential equations:

$$\left.\begin{aligned}
&[(\mu + \alpha) D^2 - (\lambda + 2\mu)\xi^2]\,\bar{u}_r - (\lambda + \mu - \alpha)\,\xi\, D\,\bar{u}_z - 2\alpha\, D\bar{\omega}_\varphi = 0 \\
&(\lambda + \mu - \alpha)\,\xi\, D\,\bar{u}_r + [(\lambda + 2\mu) D^2 - (\mu + \alpha)\xi^2]\,\bar{u}_z + 2\alpha\,\xi^2\bar{\omega}_\varphi = 0 \\
&2\alpha\, D\,\bar{u}_r + 2\,\alpha\,\xi\,\bar{u}_z + [(\gamma + \epsilon)(D^2 - \xi^2) - 4\,\alpha]\,\bar{\omega}_\varphi = 0
\end{aligned}\right\} \qquad \text{...(3.3)}$$

where $\bar{u}_r$, $\bar{u}_z$ and $\bar{\omega}_\varphi$ are the Hankel transforms of the functions u_r, u_z and ω_φ respectively and are given by

$$(\bar{u}_r, \bar{\omega}_\varphi) = \int_0^\infty (u_r, \omega_\varphi)\,\xi\, J_1(\xi r)\, dr$$

$$\bar{u}_z = \int_0^\infty u_z\,\xi\, J_0(\xi r)\, dr$$

and

$$D \equiv \frac{d}{dz},\ D^2 \equiv \frac{d^2}{dz^2}.$$

In the matrix rotations eqns. (3.3) may be represented as

$$\begin{bmatrix} \mu+\alpha & 0 & 0 \\ 0 & \lambda+2\mu & 0 \\ 0 & 0 & \gamma+\epsilon \end{bmatrix} D^2 \begin{bmatrix} \bar{u}_r \\ \bar{u}_z \\ \bar{\omega}_\varphi \end{bmatrix} - \begin{bmatrix} 0 & (\lambda+\mu-\alpha)\xi & 2\alpha \\ -(\lambda+\mu-\alpha)\xi & 0 & 0 \\ -2\alpha & 0 & 0 \end{bmatrix}$$

(*equation continued on p.* 1242)

$$D\begin{bmatrix} \bar{u}_r \\ \bar{u}_z \\ \bar{\omega}_\varphi \end{bmatrix} - \begin{bmatrix} (\lambda + 2\mu)\,\xi^2 & 0 & 0 \\ 0 & (\mu+\alpha)\,\xi^2 & -2\alpha\xi \\ 0 & -2\alpha\,\xi & (\gamma+\epsilon)\xi^2 + 2\alpha \end{bmatrix}\begin{bmatrix} \bar{u}_r \\ \bar{u}_z \\ \bar{\omega}_\varphi \end{bmatrix} = 0 \qquad \ldots(3.4)$$

we write

$$M = \begin{bmatrix} \mu+\alpha & 0 & 0 \\ 0 & \lambda+2\mu & 0 \\ 0 & 0 & \gamma+\epsilon \end{bmatrix}, N = \begin{bmatrix} 0 & (\lambda+\mu-\alpha)\xi & 2\alpha \\ -(\lambda+\mu-\alpha)\xi & 0 & 0 \\ -2\alpha & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} \bar{u}_r \\ \bar{u}_z \\ \bar{\omega}_\varphi \end{bmatrix}, X' = D\begin{bmatrix} \bar{u}_r \\ \bar{u}_z \\ \bar{\omega}_\varphi \end{bmatrix}$$

$$P = \begin{bmatrix} (\lambda+2\mu)\,\xi^2 & 0 & 0 \\ 0 & (\mu+\alpha)\,\xi^2 & -2\alpha\xi \\ 0 & -2\alpha\xi & (\gamma+\epsilon)\,\xi^2 + 4\alpha \end{bmatrix}, X'' = D^2\begin{bmatrix} \bar{u}_r \\ \bar{u}_z \\ \bar{\omega}_\varphi \end{bmatrix}$$

$$B_1 = M^{-1} N$$

and

$$B_2 = M^{-1} P. \qquad \ldots(3.5)$$

Equation (3.4) with the aid of (3.5) reduces to the form

$$X'' = B_1 X' + B_2 X. \qquad \ldots(3.6)$$

Using block matrices eqn. (3.6) assumes the form

$$\frac{d}{dz}\begin{bmatrix} X' \\ X \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ I & 0 \end{bmatrix}\begin{bmatrix} X' \\ X \end{bmatrix} \qquad \ldots(3.7)$$

where I is a 3×3 unit matrix.

Writing

$$A = \begin{bmatrix} B_1 & B_2 \\ I & 0 \end{bmatrix} \text{ and } \mathbf{V} = \begin{bmatrix} X' \\ X \end{bmatrix} \qquad \ldots(3.8)$$

eqn. (3.7) assumes the form

$$\frac{d\mathbf{V}}{dz} = A\mathbf{V}. \qquad \ldots(3.9)$$

Assume $\mathbf{V} = Y \exp(t\,z)$ to be a solution of eqn. (3.9). Then we must have

$$A\,Y = t\,Y. \qquad \ldots(3.10)$$

This gives that t is an eigenvalue of the matrix A and Y the corresponding eigenvector.

The eigeuvalues of the matrix A are the roots of

$$\det (A - t\, I) = 0. \qquad \text{...(3.11)}$$

Using simplified notations, eqn. (3.11) may be written explicitly as

$$\begin{vmatrix} -t & a_{12} & a_{13} & a_{14} & 0 & 0 \\ a_{21} & -t & 0 & 0 & a_{25} & a_{26} \\ a_{31} & 0 & -t & 0 & a_{35} & a_{36} \\ 1 & 0 & 0 & -t & 0 & 0 \\ 0 & 1 & 0 & 0 & -t & 0 \\ 0 & 0 & 1 & 0 & 0 & t \end{vmatrix} = 0 \qquad \text{...(3.12)}$$

where

$$a_{12} = \frac{\lambda + \mu - \alpha}{\mu + \alpha}\, \xi, \quad a_{13} = \frac{2\, \alpha}{\mu + \alpha}, \quad a_{14} = \frac{\lambda + 2\mu}{\mu + \alpha}\, \xi^2$$

$$a_{21} = \frac{-\lambda + \mu - \alpha}{\lambda + 2\mu}\, \xi, \quad a_{25} = \frac{\mu + \alpha}{\lambda + 2\mu}\xi^2, \quad a_{26} = \frac{-2\alpha\, \xi}{\lambda + 2\mu}$$

$$a_{31} = \frac{-\, 2\alpha}{\gamma + \epsilon} \qquad a_{35} = \frac{-2\alpha\, \xi}{\gamma + \epsilon}, \quad a_{36} = \frac{(\gamma + \epsilon)\xi^2 + 4\alpha}{\gamma + \epsilon}. \qquad \text{...(3.13)}$$

Simplifying (3.12) and using (3.13) therein we get the characteristic equation as

$$t^6 - (2\xi^2 + \zeta^2)\, t^4 + (\xi^4 + 2\xi^2\, \zeta^2)\, t^2 - \xi^4\, \zeta^2 = 0 \qquad \text{...(3.14)}$$

where

$$\zeta^2 = \xi^2 + m^2$$

and

$$m^2 = \frac{4\, \alpha\, \mu}{(\mu + \alpha)\, (\gamma + \epsilon)}.$$

The roots of (3.14) are

$$\xi,\ \xi,\ -\ \xi,\ -\ \xi,\ \zeta,\ -\ \zeta.$$

The eigenvalues of the matrix A are the root of the equation (3.14). Write $t_1 = \xi$, $t_2 = -\ \xi$, $t_3 = \zeta$, and $t_4 = -\ \zeta$. The four eigenvectors corresponding to four distinct eigenvalues t_1, t_2, t_3, t_4 of the matrix A are obtained by solving the following homogeneous equations

$$\begin{bmatrix} -t & a_{12} & a_{13} & a_{14} & 0 & 0 \\ a_{21} & -t & 0 & 0 & a_{25} & a_{26} \\ a_{31} & 0 & -t & 0 & a_{35} & a_{36} \\ 1 & 0 & 0 & -t & 0 & 0 \\ 0 & 1 & 0 & 0 & -t & 0 \\ 0 & 0 & 1 & 0 & 0 & -t \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \\ y_5(t) \\ y_6(t) \end{bmatrix} = 0 \qquad \text{...(3.15)}$$

for $t = t_1, t_2, t_3\ t_4$.

Denote by $A_1(t), A_2(t), \ldots, A_6(t)$ the co-factors of the elements of the first row of the coefficient matrix in eqn. (3.15), then

$$Y(t) = \begin{bmatrix} A_1(t) \\ A_2(t) \\ A_3(t) \\ A_4(t) \\ A_5(t) \\ A_6(t) \end{bmatrix} \qquad \text{...(3.16)}$$

are the solutions of eqn. (3.15) and hence they are eigenvectors corresponding to the eigenvalues t_1, t_2, t_3, t_4 of the matrix A. By actual calculations we obtain

$$A_1(t) = -t^5 + t^4\left(\frac{\lambda + 3\mu + \alpha}{\lambda + 2\mu}\xi^2 + \frac{4\alpha}{\gamma + \epsilon}\right)$$

$$-t\left(\frac{\mu + \alpha}{\lambda + 2\mu}\xi^4 + \frac{4\alpha\mu\xi^2}{(\lambda + 2\mu)(\gamma + \epsilon)}\right)$$

$$A_2(t) = t^2\xi\left[\frac{\lambda + \mu - \alpha}{\lambda + 2\mu}(t^2 - \xi^2) - \frac{4\alpha(\lambda + \mu)}{(\lambda + 2\mu)(\gamma + \epsilon)}\right]$$

$$A_3(t) = \frac{2\alpha t^2(t^2 - \xi^2)}{(\gamma + \epsilon)}$$

$$A_4(t) = -t^4 + t^2\left[\frac{\lambda + 3\mu + \alpha}{\lambda + 2\mu}\xi^2 + \frac{4\alpha}{(\gamma + \epsilon)}\right] - \xi^2\zeta^2\frac{\mu + \alpha}{\lambda + 2\mu}$$

$$A_5(t) = \left\{\frac{\lambda + \mu - \alpha}{\lambda + 2\mu}(t^2 - \xi^2) - \frac{4\alpha(\lambda + \mu)}{(\lambda + 2\mu)(\gamma + \epsilon)}\right\}t\xi$$

$$A_6(t) = \frac{2\alpha}{\gamma + \epsilon}t(t^2 - \xi^2). \qquad \text{...(3.17)}$$

Since t_1 and t_2 are double root of the characteristic equation (3.14) of the matrix A, the solution of the differential equation (3.9) is given by (vide Das *et al.*[2]).

$$V = C_1 Y(t_1) \exp(t_1 z) + C_2\, d/dz\, [Y(t) \exp(t\, z)]_{t=t_1}$$
$$+ C_3\, y(t_2) \exp(t_2 z) + C_3\, d/dz\, [Y(t) \exp(tz)]_{t=t_2}$$
$$+ C_5\, Y(t_3) \exp(t_3 z) + C_6\, Y(t_4) \exp(t_4 z) \qquad \ldots(3.18)$$

where $C_1, C_2, \ldots, C_6$ are arbitrary constants to be determined from boundary conditions.

Equation (3.18) can be rewritten as

$$V = (C_1 + C_2 z)\, Y(t_1) \exp(t_1 z) + C_2\, \dot{Y}(t_1) \exp(t_1 z)$$
$$+ (C_3 + C_4 z)\, Y(t_2) \exp(t_2 z) + C_4\, \dot{Y}(t_2) \exp(t_2 z)$$
$$+ C_5\, Y(t_3) \exp(t_3 z) + C_6\, Y(t_4) \exp(t_4 z) \qquad \ldots(3.19)$$

where dot (.) represents the differentiation with respect to t.

For the half space $z \geqslant 0$, equation (3.19) reduces the form

$$V = (C_3 + C_4 z)\, Y(t_2 z) \exp(t_2 z) + C_4\, \dot{Y}(t_2) \exp(t_2 z)$$
$$+ C_6\, Y(t_4) \exp(t_4 z) \qquad \ldots(3.20)$$

where the constants C_3, C_4 and C_6 are to be determined from the boundary conditions.

Equations (3.20) can be written explicitly as

$$\bar{u}'_r(z) = \{(C_3 + C_4 z)\, A_1(t_2) + C_4\, \dot{A}_1(t_2)\} \exp(t_2 z)$$
$$+ C_6\, A_1(t_4) \exp(t_4 z)$$

$$\bar{u}'_z(z) = \{(C_3 + C_4 z)\, A_2(t_2) + C_4\, \dot{A}_2(t_2)\} \exp(t_2 z)$$
$$+ C_6\, A_2(t_4) \exp(t_4 z)$$

$$\bar{\omega}'_\varphi(z) = \{(C_3 + C_4 z)\, A_3(t_2) + C_4\, \dot{A}_3(t_2)\} \exp(t_2 z)$$
$$+ C_6\, A_3(t_4) \exp(t_4 z)$$

$$\bar{u}_r(z) = \{(C_3 + C_4 z)\, A_4(t_2) + C_4\, \dot{A}_4(t_2)\} \exp(t_2 z)$$
$$+ C_6\, A_4(t_4) \exp(t_4 z)$$

$$\bar{u}_z(z) = \{(C_3 + C_4 z)\, A_5(t_2) + C_4\, \dot{A}_5(t_2)\} \exp(t_2 z)$$
$$+ C_6\, A_5(t_4) \exp(t_4 z)$$

$$\bar{\omega}_\varphi(z) = \{(C_3 + C_4 z)\, A_6(t_2) + C_4\, \dot{A}_6(t_2)\} \exp(t_2 z)$$
$$+ C_6\, A_6(t_4) \exp(t_4 z) \qquad \ldots(3.21)$$

where C_3, C_4 and C_6 are arbitrary constants to be determined from the boundary conditions. Thus the displacements have been obtained in the transformed domain and as such the stresses can also be obtained from (3.2) using (3.21) in the transformed domain.

4. Solution of Equations (2.7)

Here we are concerned with the set of equations (2.7) in which the displacement vector $\mathbf{u} = (0, u_\varphi, 0)$ and the rotation vector $\omega = (\omega_r, 0, \omega_z)$. The field of displacements $(0, u_\varphi, 0)$ and rotations $(\omega_r, 0, \omega_z)$ described by the set of equations (2.7) induces the following state of force-stress and couple-stress (vide, Dhaliwal)

$$\sigma = \begin{vmatrix} 0 & \sigma_{r\varphi} & 0 \\ \sigma_{\varphi r} & 0 & \sigma_{\varphi z} \\ 0 & \sigma_{z\varphi} & 0 \end{vmatrix} \text{ and } \mu = \begin{vmatrix} \mu_{rr} & 0 & \mu_{rz} \\ 0 & \tau_{\varphi\varphi} & 0 \\ \mu_{zr} & 0 & \mu_{zz} \end{vmatrix} \qquad \ldots(4.1)$$

where

$$\sigma_{r\varphi} = (\mu + \alpha) \frac{\partial u_\varphi}{\partial \gamma} - (\mu - \alpha) \frac{u_\varphi}{r} - 2\alpha\, \omega_z$$

$$\sigma_{\varphi r} = (\mu - \alpha) \frac{\partial u_\varphi}{\partial r} - (\mu + \alpha) \frac{u_\varphi}{r} + 2\alpha\, \omega_z$$

$$\sigma_{\varphi z} = (\mu - \alpha) \frac{\partial u_\varphi}{\partial z} - 2\alpha\, \omega_r$$

$$\sigma_{z\varphi} = (\mu + \alpha) \frac{\partial u_\varphi}{\partial z} + 2\alpha\, \omega_r$$

$$\mu_{rr} = \beta\chi + 2\gamma \frac{\partial \omega_r}{\partial r}, \quad \mu_{zz} = \beta\chi + 2r \frac{\partial \omega_z}{\partial z}, \qquad \mu_{\varphi\varphi} = \beta\chi + 2\gamma\, \omega_r$$

$$\mu_{rz} = (\gamma - \epsilon) \frac{\partial \omega_r}{\partial z} + (\gamma + \epsilon) \frac{\partial \omega_z}{\partial r}, \quad \mu_{zr} = (\gamma + \epsilon) \frac{\partial \omega_z}{\partial z}$$
$$+ (\gamma - \epsilon) \frac{\partial \omega_z}{\partial r}. \qquad \ldots(4.2)$$

Now Hankel transform of the set of equations (2.7) give

$$\left.\begin{aligned} &[(\gamma + \epsilon) D^2 - (\beta + 2\gamma)\xi^2 - 4\alpha]\,\bar{\omega}_r - (\beta + \gamma - \epsilon)\,\xi D\,\bar{\omega}_z \\ &\qquad - 2\alpha D\,\bar{u}_\varphi = 0 \\ &(\beta + \gamma - \epsilon)\,\xi D\,\bar{\omega}_r + [(\beta + 2\gamma) D^2 - (\gamma + \epsilon)\xi^2 - 4\alpha]\,\bar{\omega}_z \\ &\qquad + 2\alpha\xi\,\bar{u}_\varphi = 0 \\ &2\alpha D\,\bar{\omega}_r + 2\alpha\xi\,\bar{\omega}_z + (\mu + \alpha)(D^2 - \xi^2)\,\bar{u}_\varphi = 0. \end{aligned}\right\} \qquad \ldots(4.3)$$

Now as in section 3 equations (4.3) can be written as vector-matrix differential equation form as

$$\frac{d}{dz}(\mathbf{V}) = B\,\mathbf{V} \qquad \ldots(4.4)$$

where

$$\mathbf{V} = \begin{bmatrix} \bar{\omega}_r' \\ \bar{\omega}_z' \\ \bar{u}_\varphi' \\ \bar{\omega}_r \\ \bar{\omega}_z \\ \bar{u}_\varphi \end{bmatrix} \qquad B = \begin{bmatrix} 0 & b_{12} & b_{13} & b_{14} & 0 & 0 \\ b_{21} & 0 & 0 & 0 & b_{25} & b_{26} \\ b_{31} & 0 & 0 & 0 & b_{35} & b_{36} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad \ldots(4.5)$$

and

$$b_{12} = \frac{\beta+\gamma-\epsilon}{\gamma+\epsilon}\,\xi, \quad b_{13} = \frac{2\alpha}{\gamma+\epsilon}, \quad b_{14} = \frac{(\beta+2\gamma)\,\xi^2-4\alpha}{\gamma+\epsilon}$$

$$b_{21} = -\frac{\beta+\gamma-\epsilon}{\beta+2\gamma}\,\xi, \quad b_{25} = \frac{(\gamma-\epsilon)\,\xi^2+4\alpha}{\beta+2\gamma}, \quad b_{26} = \frac{-2\alpha\,\xi}{\beta+2\gamma}$$

$$b_{31} = \frac{-2\alpha}{\mu+\alpha}, \quad b_{35} = \frac{-2\alpha\,\xi}{\mu+\alpha}, \quad b_{36} = \xi^2 \qquad \ldots(4.6)$$

Assume $\mathbf{V} = \mathbf{X}\exp(t\,z)$ be the solution of the equation (4.4). Then we must have

$$B\,\mathbf{X} = t\,\mathbf{X}. \qquad \ldots(4.7)$$

This gives that t is an eigenvalue of the matrix B and X the corresponding eigenvectors. The eigenvalues for the matrix B are the roots of

$$\det(B - t\,I) = 0 \qquad \ldots(4.8)$$

That is,

$$\begin{vmatrix} -t & b_{12} & b_{13} & b_{14} & 0 & 0 \\ b_{21} & -t & 0 & 0 & b_{25} & b_{26} \\ b_{31} & 0 & -t & 0 & b_{35} & b_{36} \\ 1 & 0 & 0 & -t & 0 & 0 \\ 0 & 1 & 0 & 0 & -t & 0 \\ 0 & 0 & 1 & 0 & 0 & -t \end{vmatrix} = 0 \qquad \ldots(4.9)$$

where b_{12}, b_{13}, etc. are given by (4.6).

Now simplifying (4.9) and using (4.6) therein we obtain the characteristic equation as

$$t^6 - t^4 \left(\xi^2 + \lambda_1^2 + \lambda_2^2\right) + t^2 \left(\xi^2 \lambda_1^2 + \xi^2 \lambda_2^2 + \lambda_1^2 \lambda_2^2\right) - \xi^2 \lambda_1^2 \lambda_2^2 = 0 \quad \ldots(4.10)$$

where

$$\lambda_1^2 = \xi^2 + K_1^2, \quad \lambda_2^2 = \xi^2 + K_2^2$$

$$K_1^2 = \frac{4\alpha}{\beta + 2\gamma}, \quad K_2^2 = \frac{4\alpha\mu}{(\mu + \alpha)(\gamma + \epsilon)}$$

whose roots are $\xi, -\xi, \lambda_1, -\lambda_1, \lambda_2, -\lambda_2$, which are the distinct eigenvalues of the matrix B. The corresponding eigenvectors are obtained by solving the following homogeneous equation.

$$\begin{bmatrix} -t & b_{12} & b_{13} & b_{14} & 0 & 0 \\ b_{21} & -t & 0 & 0 & b_{25} & b_{26} \\ b_{31} & 0 & -t & 0 & b_{35} & b_{36} \\ 1 & 0 & 0 & -t & 0 & 0 \\ 0 & 1 & 0 & 0 & -t & 0 \\ 0 & 0 & 1 & 0 & 0 & -t \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} = 0 \quad \ldots(4.11)$$

for $t = t_i$, $i = 1, 2, \ldots, 6$ where $t_1 = \xi$, $t_2 = -\xi$, $t_3 = \lambda_1$, $t_4 = -\lambda_1$, $t_5 = \lambda_2$, $t_6 = -\lambda_2$.

Denote by $B_i(t)$, $i = 1, 2, \ldots; 6$, the co-factors of the elements of the first row of the coefficient matrix in eqn. (4.11).

Then

$$X(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \\ B_5(t) \\ B_6(t) \end{bmatrix}$$

are the solutions of the eqn. (4.11) and hence they are eigenvectors corresponding to the eigenvalues t_i, $i = 1, 2, \ldots, 6$, of the matrix B. By actual calculation we obtain

$$B_1(t) = -t^5 + t^2 \left\{ \frac{\xi^2(\beta + 3\gamma + \epsilon) + 4\alpha}{\beta + 2\gamma} \right\}$$

(*equation continued on p.* 1249)

$$- t\,\xi^2 \left\{ \frac{(\mu + \alpha)(\gamma + \epsilon)\,\xi^2 + 4\alpha\,\mu}{(\beta + 2\gamma)(\mu + \alpha)} \right\}$$

$$B_2(t) = - \frac{(\beta + \gamma - \epsilon)}{(\beta + 2\gamma)} t^4\,\xi - t^2\,\xi \left\{ \frac{4\,\alpha^2 - (\mu + \alpha)(\beta + \gamma - \alpha\xi^2)}{(\beta + 2\gamma)(\mu + \alpha)} \right\}$$

$$B_3(t) = - t^4 \frac{2\alpha}{\mu + \alpha} - t^2 \frac{2\alpha\,\xi^2\,(\beta + \gamma - \epsilon)}{(\mu + \alpha)(\beta + 2\gamma)}$$

$$B_4(t) = t^4 - t^2 \left\{ \frac{(\beta + \gamma - \epsilon)\,\xi^2 - 4\alpha}{\beta + 2\gamma} \right\}$$

$$+ \xi^2 \frac{(\gamma + \epsilon)(\mu + \alpha)\,\xi^2 + 4\alpha\,\mu}{(\mu + \alpha)(\beta + 2\gamma)}$$

$$B_5(t) = t^3\,\xi \frac{\beta + \gamma - \epsilon}{\beta + 2\gamma} - t \left\{ \frac{\beta + \gamma - \epsilon}{\beta + 2\gamma}\,\xi^3 + \frac{4\,\alpha^2\,\xi}{(\beta + 2\gamma)(\mu + \alpha} \right\}$$

$$B_6(t) = - t^3 \frac{2\alpha}{\mu + \alpha} - t \left\{ \frac{2\alpha\,\xi\,(\beta + \gamma - \epsilon) - 2\alpha\,(\gamma + \epsilon)\,\xi^2 + 8\alpha^2}{(\beta + 2\gamma)(\mu + \alpha)} \right\}$$

Since $t_1, t_2, \ldots, t_6$ are all distinct roots of the characteristic equation (4.10) of the matrix B, the general solution of the differential equation (4.4) is given by (vide appendix A)

$$\mathbf{V} = E_1\,X(t_1) \exp(t_1\,z) + E_2\,X(t_2) \exp(t_2\,z) + E_3\,X(t_3) \exp(t_3\,z)$$
$$+ E_4\,X(t_4) \exp(t_4\,z) + E_5\,X(t_5) \exp(t_5\,z)$$
$$+ E_6\,X(t_6) \exp(t_6\,z)$$

where

$$E_i,\ i = 1, 2, \ldots, 6.$$

are arbitrary constants

i. e.

$$\mathbf{V} = \sum_{i=1}^{6} E_i\,X(t_i) \exp(t_i\,z)$$

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Appendix A

Consider the differential equation

$$\frac{dv}{dt} = Av \qquad \text{...(A1)}$$

where v is an n vector and A is an $n \times n$ real constant matrix.

If $\lambda_1\ \lambda_2, \ldots, \lambda_n$ are distinct eigenvalues of the matrix A and $x_1, x_2 \ldots x_n$ be the corresponding eigenvectors of A, then the general solution of (A1) is given by

$$v(t) = C_1 X_1 e^{\lambda_1 t} + \ldots + C_n X_n e^{\lambda_n t}$$

where $C_1, \ldots C_n$, are arbitrary constants.

If λ_1 is an eigenvalue of A of multiplicity 2 and all other eigenvalues $\lambda_3; \ldots \lambda_n$ are of multiplicity one and $X_1, X_3, \ldots X_n$ the corresponding eigenvector of A, then the general solution of (A1) is given by

$$v(t) = C_1 X_1 e^{\lambda_1 t} + C_2 \frac{d}{dt}(X_1 e^{\lambda_1 t}) + C_3 X_3 e^{\lambda_3 t} + \ldots + C_n X_n e^{\lambda_n t}$$

where $C_1, C_2, \ldots, C_n$ are all arbitrary constants.

For details, vide Das *et al.*[2].

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MELLIN TRANSFORM OF THE GRAVITY EFFECT OF A 2—D HORIZONTAL CIRCULAR CYLINDER WITH VARIABLE DENSITY

L. ANAND BABU[1], N. L. MOHAN[2], N. SUNDARARAJAN[2]

AND

S. V. SESHAGIRI RAO[2]

[1]*Department of Mathematics, Osmania University, Hyderabad* 500007

[2]*Centre of Exploration Geophysics, Osmania University, Hyderabad* 500007

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The Mellin transform of the gravity effect of a buried 2– D horizontal circular cylinder with density contrast varying linearly with depth is derived and analysed to extract the body parameters. The characteristic features of the Mellin transform of the gravity effect resemble gamma function. The validity of the method is tested on simulated models. The stability of the Mellin transform is studied by incorporating random noise in the gravity effect at different levels and subsequently the error estimation of the interpreted values is discussed.

INTRODUCTION

In general, the interpretation of the geophysical anomalies are carried out by a ssuming certain geometrical shapes with uniform physical properties. Very frequently, we come across some practical cases wherein for example the density contrast increases with the increase of depth which is evident from seismic studies.

Therefore, it would be meaningful to interpret such geophysical anomalies with non uniform density[4]. Herein the analysis of gravity anomalies due to a horizontal circular cylinder with the density contrast varying linearly with depth is presented using the Mellin transform. Such transformation of gravity (or magnetic) anomalies paves the way for simplified analysis of the complex potential field[1]. The procedure is illustrated with three sets of theoretical models. The stability of the Mellin transform is studied by incorporating random noise in the gravity effect at different levels and subsequently the error estimation of the interpreted values is discussed.

MELLIN TRANSFORM OF THE GRAVITY EFFECT OF THE HORIZONTAL CIRCULAR CYLINDER

A buried horizontal circular cylinder extending infinitely along the Y-direction, with its normal section parallel to the $X - Z$ plane is considered. The origin of the

coordinate system is taken on the ground surface such that the Z-axis coincides with the diameter (Fig. 1a). Let the density contrast at the apex of the cylinder as ρ and the rate of change of density contrast varying linearly with depth be a. In this case, the gravity effect of the cylinder is given by Radhakrishan Murthy[5].

$$g(x) = A \frac{Z}{X^2 + Z^2} - B \frac{Z^2 - X^2}{(X^2 + Z^2)^2} \quad \text{...(1)}$$

where

$$A = 2\pi \, GR^2 \, (\rho + aR) \quad \text{...(1a)}$$

$$B = \frac{\pi \, GaR^4}{2}. \quad \text{...(1b)}$$

Z is the depth to the centre of the cylinder, R the radius of the cylinder and G the universal gravitational constant.

The Mellin transform of the gravity effect given by eqn. (1) is written as Sneddon[6]

$$M(s) = \int_0^\infty x^{s-1} \, g(x) \, dx \quad \text{...(2)}$$

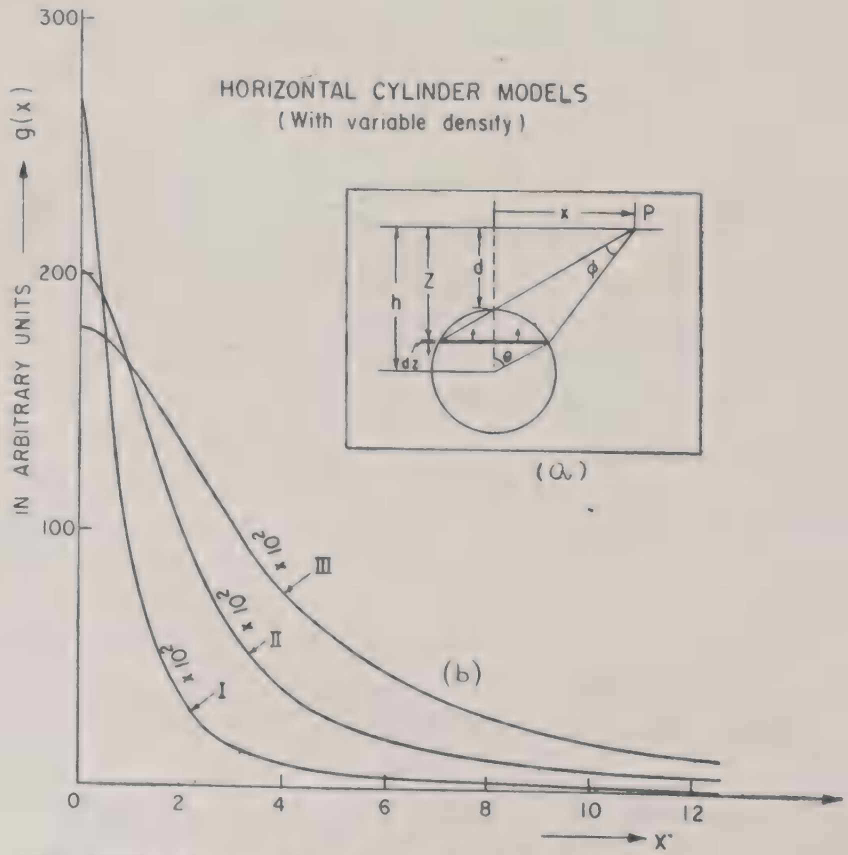

FIG. 1 (a) Cross section of the buried horizontal circular cylinder.
(b) Computed gravitational effect of horizontal circular cylinder

where s is a real positive integer or fractional number.

Substituting for $g(x)$ and integrating eqn. (2) with respect to x, we get the Mellin transform of $g(x)$ as :

$$M(s) = [A\, Z^{s-1}\, \Gamma(s/2)\, \Gamma((2-s)/2) - BZ^{s-2} \times \{\Gamma(s/2)\, \Gamma((4-s)/2) - \Gamma((s+2)/2)\, \Gamma((2-s)/2)\}] \quad \ldots (3)$$

$$(0 < s < 2)$$

ANALYSIS

For a set of arbitrary values of s, i, e., for $s = 1/4$; $s = 1/2$; $s = 3/4$; $s = 1$ and $s = 5/4$ equation (3) is written as :

$$M(1/4) = A\, Z^{-3/4}\, \Gamma(1/8)\, \Gamma(7/8) - B\, Z^{-7/4}\, [\Gamma(1/8)\, \Gamma(15/8) - \Gamma(9/8)\, \Gamma(7/8)] \quad \ldots (4a)$$

$$M(1/2) = A\, Z^{-1/2}\, \Gamma(1/4)\, \Gamma(3/4) - B\, Z^{3/2}\, [\Gamma(1/4)\, \Gamma(7/4) - \Gamma(5/4)\, \Gamma(3/4)] \quad \ldots (4b)$$

$$M(3/4) = A\, Z^{1/4}\, \Gamma(3/8)\, \Gamma(5/8) - B\, Z^{5/4}\, [\Gamma(3/8)\, \Gamma(13/8) - \Gamma(11/8)\, \Gamma(5/8)] \quad \ldots (4c)$$

$$A = M(1)/\pi \quad \ldots (4d)$$

$$M(5/4) = A\, Z^{1/4}\, \Gamma(5/8)\, \Gamma(3/8) - B\, Z^{-3/4}\, [\Gamma(5/8)\, \Gamma(11/8) - \Gamma(13/8)\, \Gamma(3/8)]. \quad \ldots (4e)$$

From eqns. (4c) and (4e), the value of Z is evaluated as :

$$M(3/4)\, Z^{1/4} = U - (B/Z)\, V \quad \ldots (5)$$

and

$$M(5/4)\, Z^{1/4} = U + (B/Z)\, V \quad \ldots (6)$$

i. e ,

$$P\, (Z^{1/4})^2 - U\, Z^{1/4} + Q = 0 \quad \ldots (7)$$

where

$$P = M(3/4)/2,$$

$$Q = M(5/4)/2$$

$$U = M(1)\, \Gamma(3/4)\, \Gamma(5/8)/\pi,$$

$$V = \Gamma(3/8)\, \Gamma(13/8) - \Gamma(11/8)\, \Gamma(5/8).$$

Hence

$$Z = \left\{ \frac{U \pm (U^2 - 4PQ)^{1/2}}{2P} \right\}^4. \quad \ldots (8)$$

Since A and Z are known, B can be evaluated as :

$$B = Z\left[\frac{M(5/4)\,Z^{-1/4} - M(3/4)\,Z^{1/4}}{2V}\right]. \qquad \ldots(9)$$

By eliminating 'a' from equations (1a) and (1b) a cubic equation in 'R' is obtained as :

$$R^3 - (A/2\,\rho\,\pi G)\,R + (4B/2\,\rho\,\pi G) = 0. \qquad (10)$$

Applying the well known Cardon's method, R is evaluated, and subsequently a is calculated as :

$$a = 2B/R^4. \qquad \ldots(11)$$

Discrete Mellin Transform

Since the gravity data is collected at discrete intervals in the real field situation, the numerical computation of the Mellin transform is carried out by formulating the discrete Mellin transform as[3]

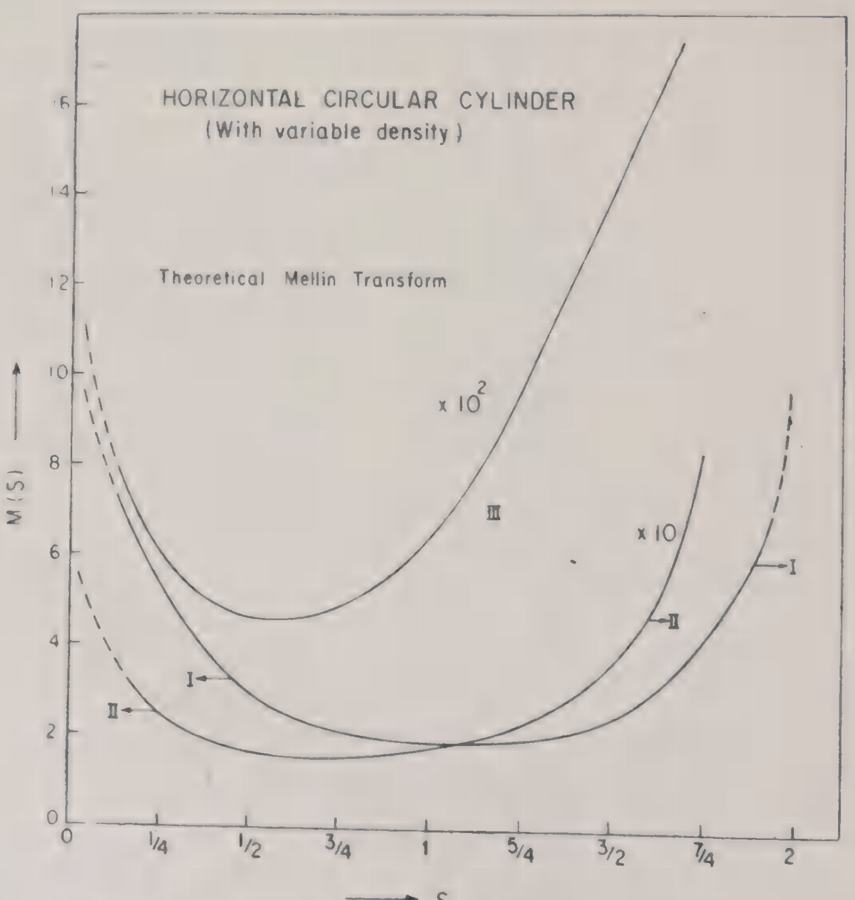

FIG. 2. Continuous Mellin transform of the gravity effect of horizontal circular cylinder with density variation.

$$M(l.\Delta s) = \sum_{l=1}^{N-1} g(n.\Delta x)\,(n.\Delta x)^{l.\Delta s-1}\,\Delta x \quad \ldots(12)$$

$$(0 < l.\Delta s < 2)$$

where $\Delta s = 1/4$, N is the total number of points and Δx is the sample interval.

Synthetic Examples

The procedure detailed in the text is illustrated with three theoretical models (Table I). The theoretical Mellin transform of the gravity effect of three models are computed using eqn (3) and shown in Fig 2.

The gravity effect due to horizontal circular cylinder with variable density for three models are computed using eqn. (1). Since the gravity effect due to horizontal circular cylinder is symmetric, only the positive side of the anomaly is shown in Fig. 1b. The discrete Mellin transform of the gravity effect of the cylindrical models are computed using eqn. (12) and shown in Fig. 3. It may be observed that the discrete Mellin

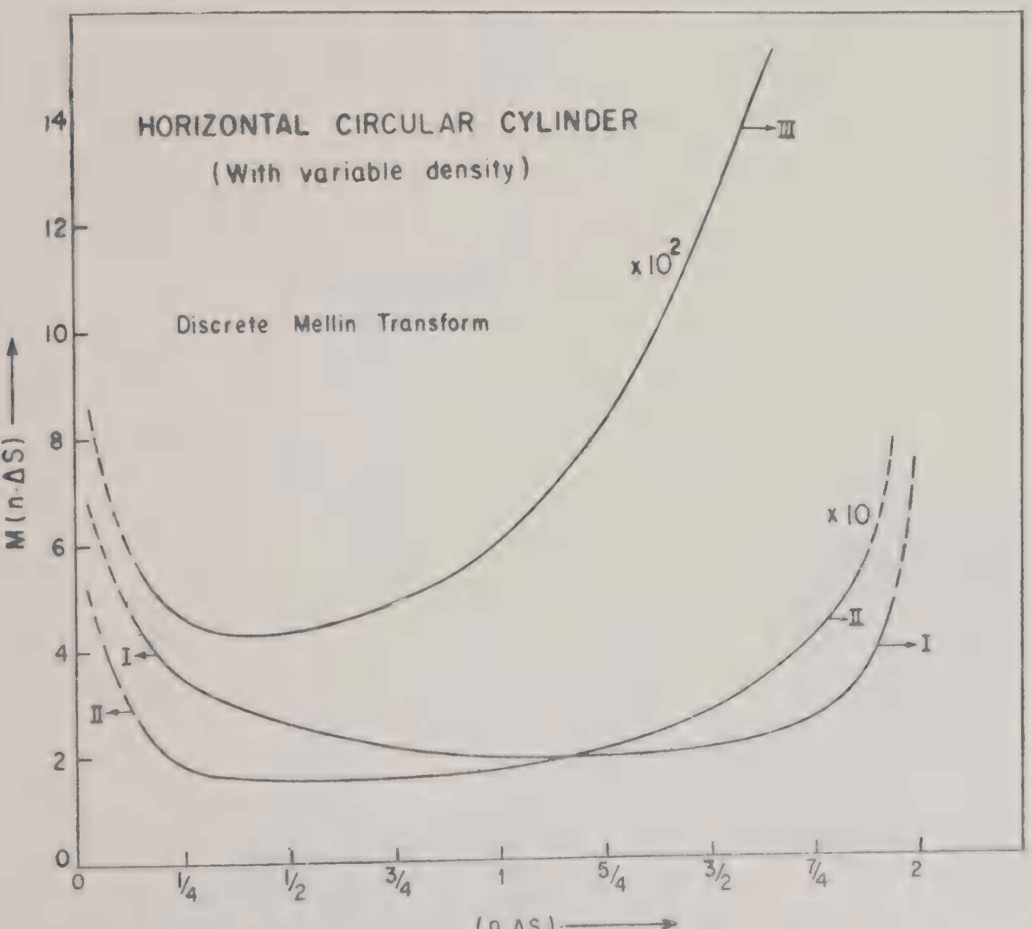

FIG. 3. Discrete Mellin transform of the gravity effect of horizontal circular cylinder with density variation.

TABLE I *Theoretical Examples*

		R^*	h^*	a^*
Model I	Assumed values	0.50	0.75	1.00
	Evaluated values	0.56	0.70	0.92
Model II	Assumed values	1.00	2.00	1.50
	Evaluated values	0.95	1.88	1.39
Model III	Assumed values	1.50	3.50	1.75
	Evaluated valves	1.45	2.95	1.80

(* in arbitrary units)

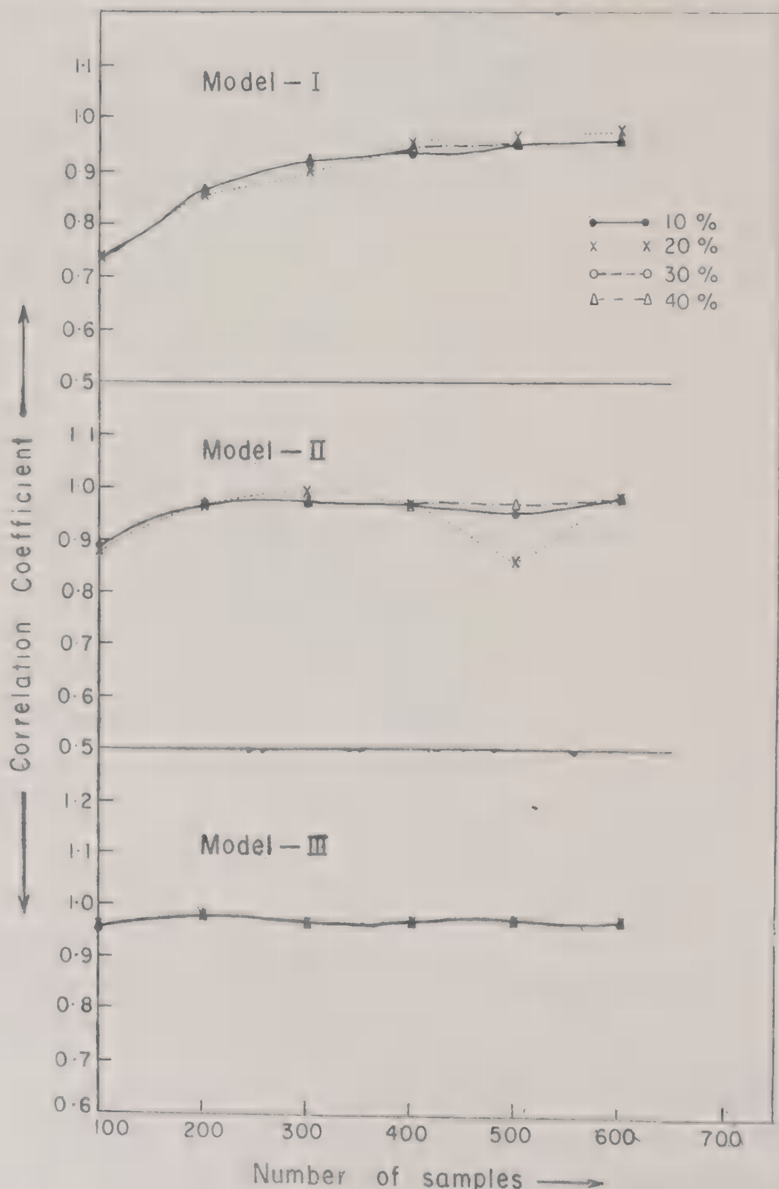

FIG. 4. Correlation coefficient versus number of discrete gravity samples.

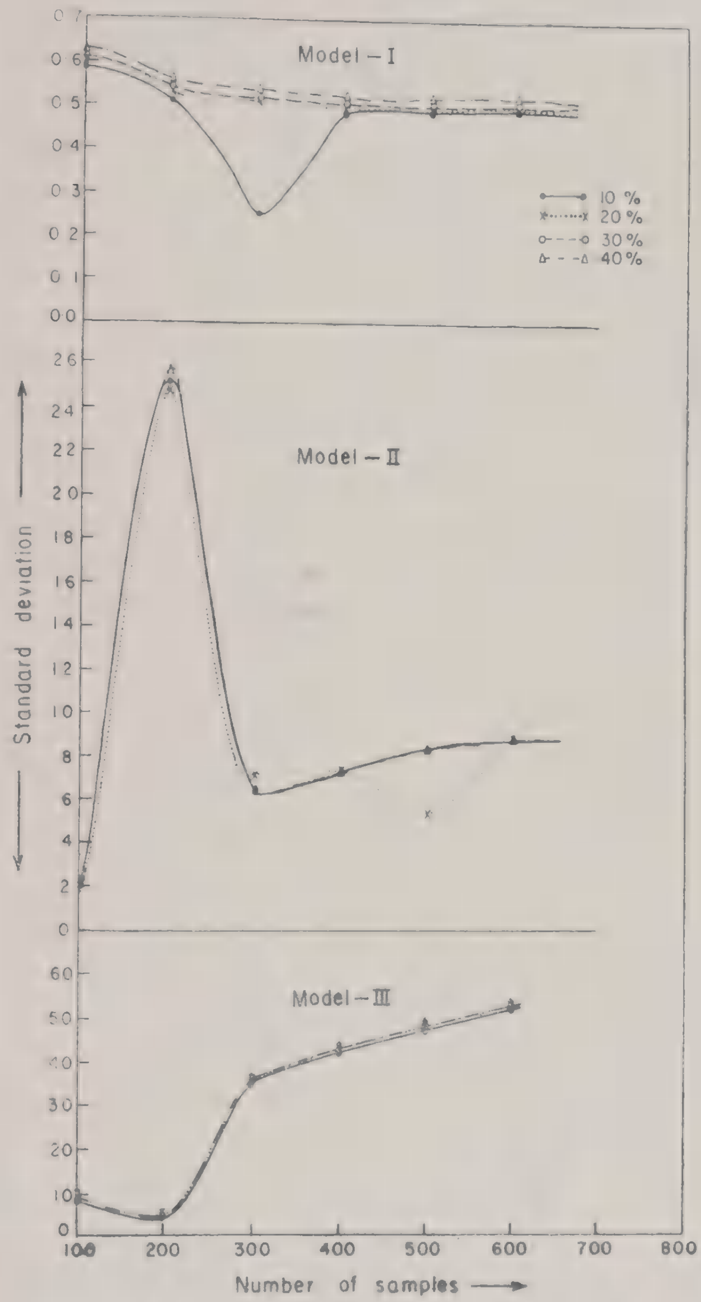

FIG. 5. Standard deviation versus number of discrete gravity samples.

transform of the gravity effect of the models (Fig. 3) are similar to the theoretical Mellin transform (Fig. 2) and they resemble gamma function curves. The parameters are evaluated from the computed discrete Mellin transform of the gravity models using eqns. (4d), (8) (10) and (11) and Fig. (3) and presented in Table 1. It may be noticed that the evaluated parameters reasonably agree with the assumed values.

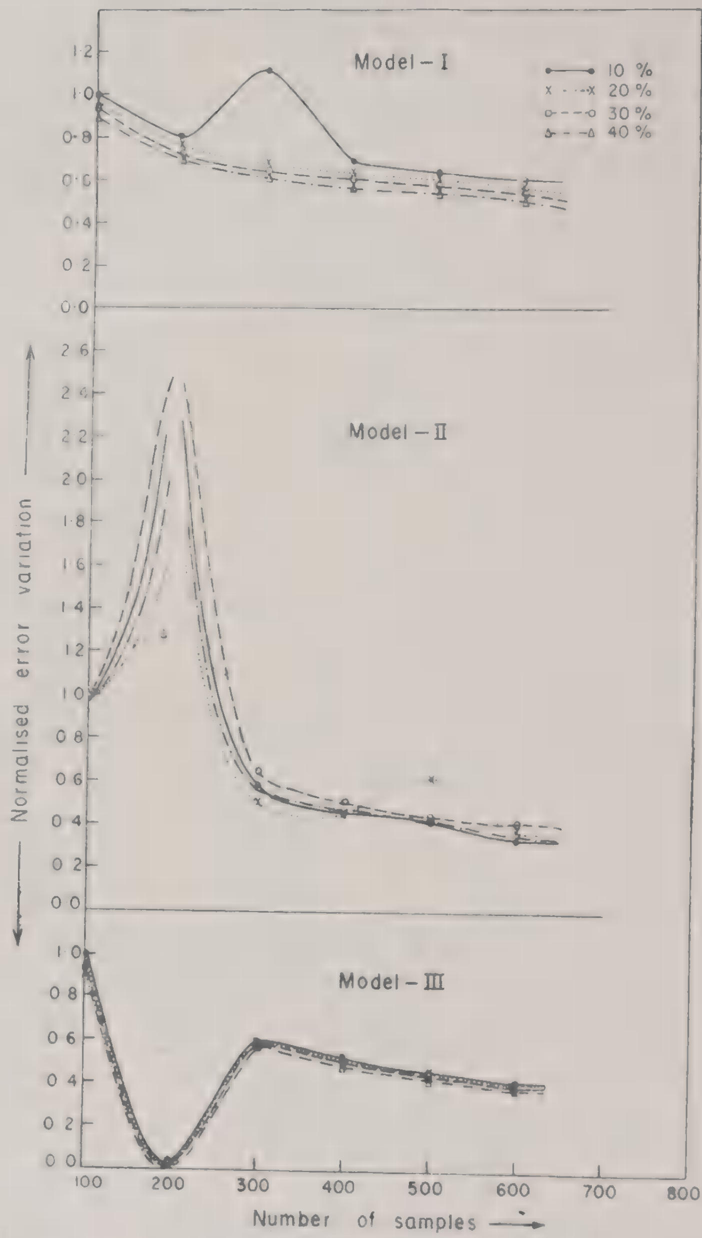

Fig. 6. Normalized error variation versus number of discrete gravity samples.

Stability Analysis

Here for all the three gravity models the number of discrete samples of the gravity effect are considered at six different levels ranging from 100 to 600 at an interval of 100. In each case the random noise with different percentages i.e., 10% 20%, 30% and 40% are added to the discrete gravity effect of the horizontal circular cylinder with variable density. The discrete Mellin transfom of the noise contaminated gravity effect is computed. Using computed theoretical Mellin transform of the gravity effect and the discrete Mellin transform of the noise contaminated gravity effect for all the

models for different levels of number of samples and with different levels of random noise, the correlation coefficient, standard deviation and the error variation (normalized) are computed and shown in Figs. 4, 5 and 6.

It is observed from the Fig. (4) (i.e.) correlation coefficient versus number of samples with different levels of noise for the gravity models, the correlation coefficient is tending to 1 for discrete gravity samples > 400. Also the standard deviation and error variation (normalized) are saturating for discrete gravity samples > 400.

The error percentage in evaluated parameters of the cylinder is relatively high ($> 30\%$) for the discrete gravity samples < 400 for different levels of random noise.

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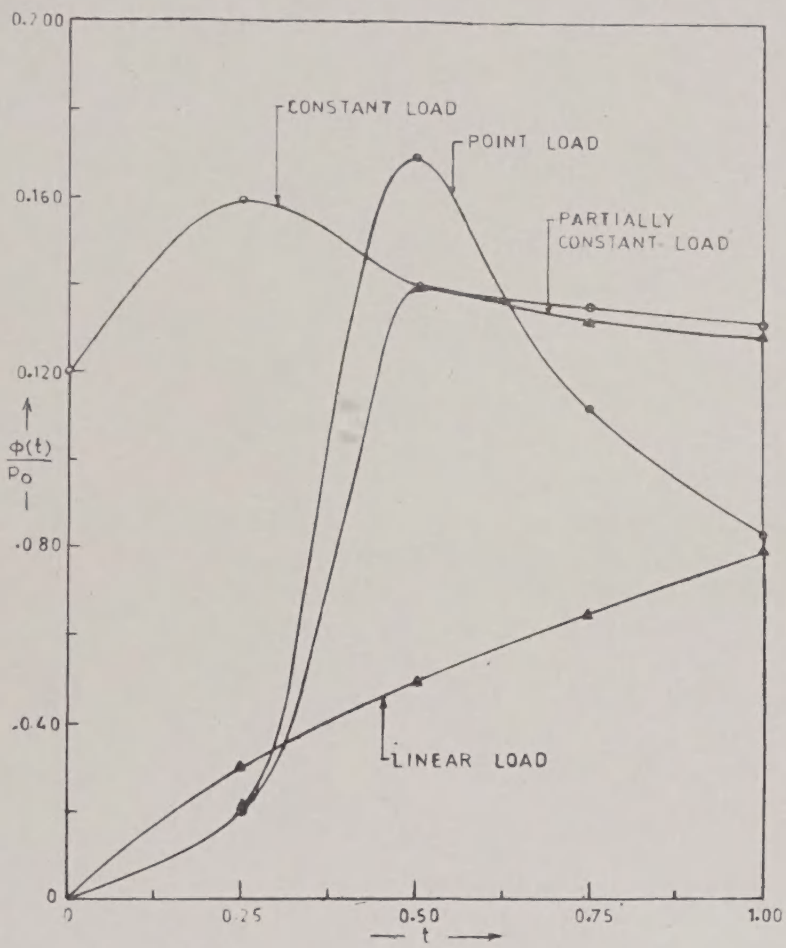

FIG. $\phi(t)/P_0$ versus t for various loading function.

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